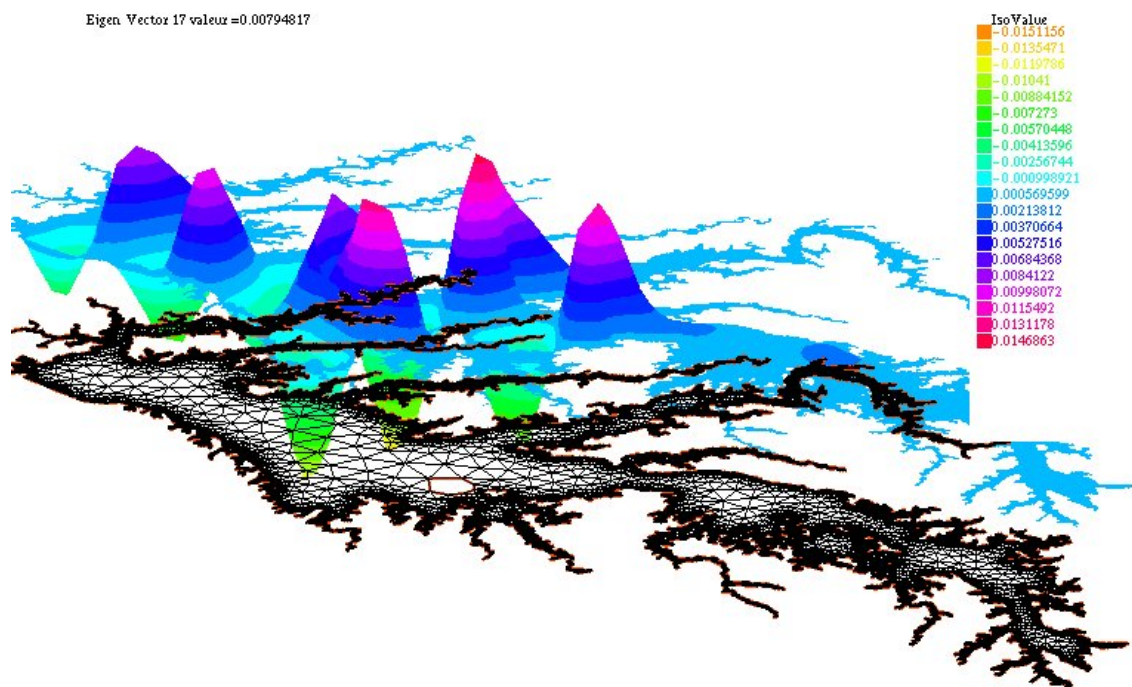


Freefem++

Third Edition, Version 3.32

<http://www.freefem.org/ff++>

F. Hecht



Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris

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Acknowledgments We are very grateful to l'École Polytechnique (Palaiseau, France) for printing the second edition of this manual (<http://www.polytechnique.fr>), and to l'Agence Nationale de la Recherche (Paris, France) for funding of the extension of FreeFem++ to a parallel tridimensional version (<http://www.agence-nationale-recherche.fr>) Référence : ANR-07-CIS7-002-01.

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Preface

ॐ पूर्णमदः पूर्णमिदं पूर्णात् पूर्णमुदच्यते ।
पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते ॥
ॐ शान्तिः शान्तिः शान्तिः ॥

Fruit of a long maturing process, freefem, in its last avatar, FreeFem++ , is a high level integrated development environment (IDE) for numerically solving partial differential equations (PDE) in dimension 2 and 3. It is the ideal tool for teaching the finite element method but it is also perfect for research to quickly test new ideas or multi-physics and complex applications.

FreeFem++ has an advanced automatic mesh generator, capable of a posteriori mesh adaptation; it has a general purpose elliptic solver interfaced with fast algorithms such as the multi-frontal method UMFPACK, SuperLU . Hyperbolic and parabolic problems are solved by iterative algorithms prescribed by the user with the high level language of FreeFem++ . It has several triangular finite elements, including discontinuous elements. Finally everything is there in FreeFem++ to prepare research quality reports: color display online with zooming and other features and postscript printouts.

This manual is meant for students at Master level, for researchers at any level, and for engineers (including financial engineering) with some understanding of variational methods for partial differential equations.

Chapter 1

Introduction

A partial differential equation is a relation between a function of several variables and its (partial) derivatives. Many problems in physics, engineering, mathematics and even banking are modeled by one or several partial differential equations.

FreeFem++ is a software to solve these equations numerically. As its name implies, it is a free software (see the copyrights for full detail) based on the Finite Element Method; it is not a package, it is an integrated product with its own high level programming language. This software runs on all UNIX OS (with g++ 3.3 or later, and OpenGL) , on Window XP, Vista and 7 and on MacOS 10 (powerpc, intel)

Moreover FreeFem++ is highly adaptive. Many phenomena involve several coupled systems, for example: fluid-structure interactions, Lorentz forces for aluminium casting and ocean-atmosphere problems are three such systems. These require different finite element approximations and polynomial degrees, possibly on different meshes. Some algorithms like Schwarz' domain decomposition method also require data interpolation on multiple meshes within one program. FreeFem++ can handle these difficulties, i.e. *arbitrary finite element spaces on arbitrary unstructured and adapted bi-dimensional meshes*.

The characteristics of FreeFem++ are:

- Problem description (real or complex valued) by their variational formulations, with access to the internal vectors and matrices if needed.
- Multi-variables, multi-equations, bi-dimensional and three-dimensional static or time dependent, linear or nonlinear coupled systems; however the user is required to describe the iterative procedures which reduce the problem to a set of linear problems.
- Easy geometric input by analytic description of boundaries by pieces; however this part is not a CAD system; for instance when two boundaries intersect, the user must specify the intersection points.
- Automatic mesh generator, based on the Delaunay-Voronoi algorithm; the inner point density is proportional to the density of points on the boundaries [7].

- Metric-based anisotropic mesh adaptation. The metric can be computed automatically from the Hessian of any FreeFem++ function [9].
- High level user friendly typed input language with an algebra of analytic and finite element functions.
- Multiple finite element meshes within one application with automatic interpolation of data on different meshes and possible storage of the interpolation matrices.
- A large variety of triangular finite elements : linear, quadratic Lagrangian elements and more, discontinuous P1 and Raviart-Thomas elements, elements of a non-scalar type, the mini-element,... (but no quadrangles).
- Tools to define discontinuous Galerkin finite element formulations P0, P1dc, P2dc and keywords: `jump`, `mean`, `intalledges`.
- A large variety of linear direct and iterative solvers (LU, Cholesky, Crout, CG, GMRES, UMFPACK, MUMPS, SuperLU, ...) and eigenvalue and eigenvector solvers (ARPACK) .
- Near optimal execution speed (compared with compiled C++ implementations programmed directly).
- Online graphics, generation of `.txt`, `.eps`, `.gnu`, `mesh` files for further manipulations of input and output data.
- Many examples and tutorials: elliptic, parabolic and hyperbolic problems, Navier-Stokes flows, elasticity, Fluid structure interactions, Schwarz's domain decomposition method, eigenvalue problem, residual error indicator, ...
- A parallel version using `mpi`

1.1 Installation

1.1.1 For everyone:

First open the following web page

`http://www.freefem.org/ff++/`

And choose your platform: Linux, Windows, MacOS X, or go to the end of the page to get the full list of downloads.

Remark 1 : *Binaries are available for Microsoft Windows, Apple Mac OS X and some Linux systems.*

Install by double click on the appropriate file, under linux and MacOS the install file are respectively in directory `/usr/local/bin`, `/usr/local/share/freefem++`, `/usr/local/lib/`

Windows binaries install First download the windows installation executable, then double click it. to install FreeFem++ . In most cases just answer yes (or typr return) to all questions. Otherwise in the Additional Task windows, check the box "Add application directory to your system path your system path ." This is required otherwise the program `ffglut.exe` will not be found.

By now you should have two new icons on your desktop:

- FreeFem++ (VERSION) .exe the FreeFem++ application.
- FreeFem++ (VERSION) Examples a link to the FreeFem++ folder of examples.

where (VERSION) is the version of the files (for example 3.3-0-P4).

By default, the installed files are in

`C:\Programs Files\FreeFem++`

In this directory, you have all the .dll files and other applications: `FreeFem++-nw.exe`, `ffglut.exe` ... the FreeFem++ application without graphic windows.

The syntax for the command-line tools are the same as those of `FreeFem.exe`.

MacOS X binaries install Download the MacOS X binary version file, extract all the files with a double click on the icon of the file, go the the directory and put the `FreeFem+.app` application in the `/Applications` directory. If you want a terminal access to FreeFem++ just copy the file `FreeFem++` in a directory of your `$PATH` shell environment variable. If you want to automatically launch the `FreeFem++.app`, double click on a .edp file icon. Under the finder pick a .edp in directory `examples++-tutorial` for example, select menu `File -> Get Info` an change `Open with:` (choose `FreeFem++.app`) and click on button `change All....`

Where to go from here An integrated environment called `FreeFem++-cs`, written by Antoine Le Hyaric, is provided with `FreeFem++` . Unless you wish to profile now your own development environment, you may proceed to the next paragraph "How to use FreeFem++".

1.1.2 For the pros: Installation from sources

This section is for those who for some reason do not wish to use the binaries and hence need to recompile `FreeFem++` or install it from the source code:

The documentation archive : The documentation is also open source; to regenerate it you need a \LaTeX environment capable of compiling a CVS archive; under MS-Windows you will have to use `mingw/msys`

<http://www.mingw.org>

and under MacOS X we have used Apple's Developer Tools "Xcode" and \LaTeX from <http://www.ctan.org/system/mac/texmac>.

The C++ archive : FreeFem++ must be compiled from the source archive, as indicated in

`http://www.freefem.org/ff++/index.htm`

To extract files from the compressed archive `freefem++-(VERSION).tar.gz` to a directory called

`freefem++-(VERSION)`

enter the following commands in a shell window :

```
tar zxvf freefem++-(VERSION).tar.gz
cd freefem++-(VERSION)
```

To compile and install FreeFem++ , just follow the INSTALL and README files. The following programs are produced, depending on the system you are running :

1. FreeFem++ , standard version, with a graphical interface based on GLUT/OpenGL (use `ffglut` visualization tool) or not just add `-nw` parameter.
2. `ffglut` the visualization tools through a pipe of `freefem++` (remark: *if `ffglut` is not in the system path, you will have no plot*)
3. FreeFem++-nw, postscript plot output only and `ffmedit` (batch version, no graphics windows via `ffglut`)
4. FreeFem++-mpi, parallel version, postscript output only
5. /Applications/FreeFem++.app, the Drag and Drop CoCoa MacOSX Application
6. `bamg` , the `bamg` mesh generator
7. `cvms2` , a mesh file convertor
8. `drawbdmesh` , a mesh file viewer
9. `ffmedit` the `freefem++` version of `medit` software (thanks to P. Frey)

The syntax of tools FreeFem++ ,FreeFem++-nw,FreeFem++-mpi, on the command-line are

- `FreeFem++ [-?] [-v nn] [-fglut file1] [-glut file2] [-f] edppath` where the
 - or `FreeFem++-nw -? [-v nn] [-fglut file1] [-glut file2] [-f] edppath` where the
- `-?` show the usage.
`-fglut filename` to store all the data for graphic in file `filename`, and to replay do `ffglut filename`.
`-glut ffglutprogram` to change the visualisator program's.

```

-nw no call to fglut and medit
-v nn set the level of verbosity to nn before execution of the script.
-ne no edp script output
-wait wait an return in text window before closing FreeFem++
-nowait wait an return in text window before closing FreeFem++
-ne no edp script output
-cd Change directory to script dir (the script path must be global)
if no file path then you get a dialog box to choose the edp file on windows systeme.

```

The notation `[]` means "optional".

Remark 2 *In most cases you can set the level of output (verbosity) to value `nn` by adding the parameters `-v nn` on the command line.*

As an installation test, under unix: go into the directory `examples++-tutorial` and run `FreeFem++` on the example script `LaplaceP1.edp` with the command :

```
FreeFem++ LaplaceP1.edp
```

If you are using `nedit` as your text editor, do one time `nedit -import edp.nedit` to have coloring syntax for your `.edp` files.

Link with other text editors

notepad++ at <http://notepad-plus.sourceforge.net/uk/site.htm>

- Open Notepad++ and Enter F5
- In the new window enter the command `launchff++ "$ (FULL_CURRENT_PATH) "`
- Click on Save, and enter `FreeFem++` in the box "Name", now choose the short cut key to launch directly `FreeFem++` (for example `alt+shift+R`)
- To add Color Syntax Compatible with `FreeFem++` In Notepad++,
 - In Menu "Parameters" -> "Configuration of the Color Syntax" proceed as follows:
 - In the list "Language" select C++
 - Add "edp" in the field "add ext"
 - Select "INSTRUCTION WORD" in the list "Description" and in the field "supplementary key word", cut and past the following list:
`P0 P1 P2 P3 P4 P5 P1dc P2dc P3dc P4dc P5dc RT0 RT1 RT2 RT3 RT4 RT5 macro plot int1d int2d solve movemesh adaptmesh trunc checkmovemesh on func buildmesh square Eigenvalue min max imag exec LinearCG NLCG Newton BFGS LinearGMRES catch try intalldges jump average mean load savemesh convect abs sin cos tan atan asin acos cotan sinh cosh tanh cotanh atanh asinh acosh pow exp log log10 sqrt dx dy endl cout`

- Select "TYPE WORD" in the list "Description" and ... " "supplementary key word", cut and past the following list
`mesh real fespace varf matrix problem string border complex ifstream of-stream`
- Click on Save & Close. Now nodepad++ is configured.

Crimson Editor available at <http://www.crimsoneditor.com/> and adapted as follows:

- Go to the Tools/Preferences/File association menu and add the .edp extension set
- In the same panel in Tools/User Tools, add a FreeFem++ item (1st line) with the path to freefem++.exe on the second line and \$(FilePath) and \$(FileDir) on third and fourth lines. Tick the 8.3 box.
- for color syntax, extract file from crimson-freefem.zip and put files in the corresponding sub-folder of Crimson folder (C:\Program Files\Crimson Editor).

winedt for Windows : this is the best but it could be tricky to set up. Download it from

<http://www.winedt.com>

this is a multipurpose text editor with advanced features such as syntax coloring; a macro is available on www.freefem.org to localize winedt to FreeFem++ without disturbing the winedt functional mode for LaTeX, TeX, C, etc. However winedt is not free after the trial period.

TeXnicCenter for Windows: this is the easiest and will be the best once we find a volunteer to program the color syntax. Download it from

<http://www.texniccenter.org/>

It is also an editor for TeX/LaTeX. It has a "tool" menu which can be configured to launch FreeFem++ programs as in:

- Select the Tools/Customize item which will bring up a dialog box.
- Select the Tools tab and create a new item: call it freefem.
- in the 3 lines below,
 1. search for FreeFem++.exe
 2. select Main file with further option then Full path and click also on the 8.3 box
 3. select main file full directory path with 8.3

nedit on the Mac OS, Cygwin/Xfree and linux, to import the color syntax do

```
nedit -import edp.nedit
```

Smultron on the Mac, available at <http://smultron.sourceforge.net>. It comes ready with color syntax for .edp file. To teach it to launch FreeFem++ files, do a "command B" (i.e. the menu Tools/Handle Command/new command) and create a command which does

```
/usr/local/bin/FreeFem++-CoCoa %%p
```

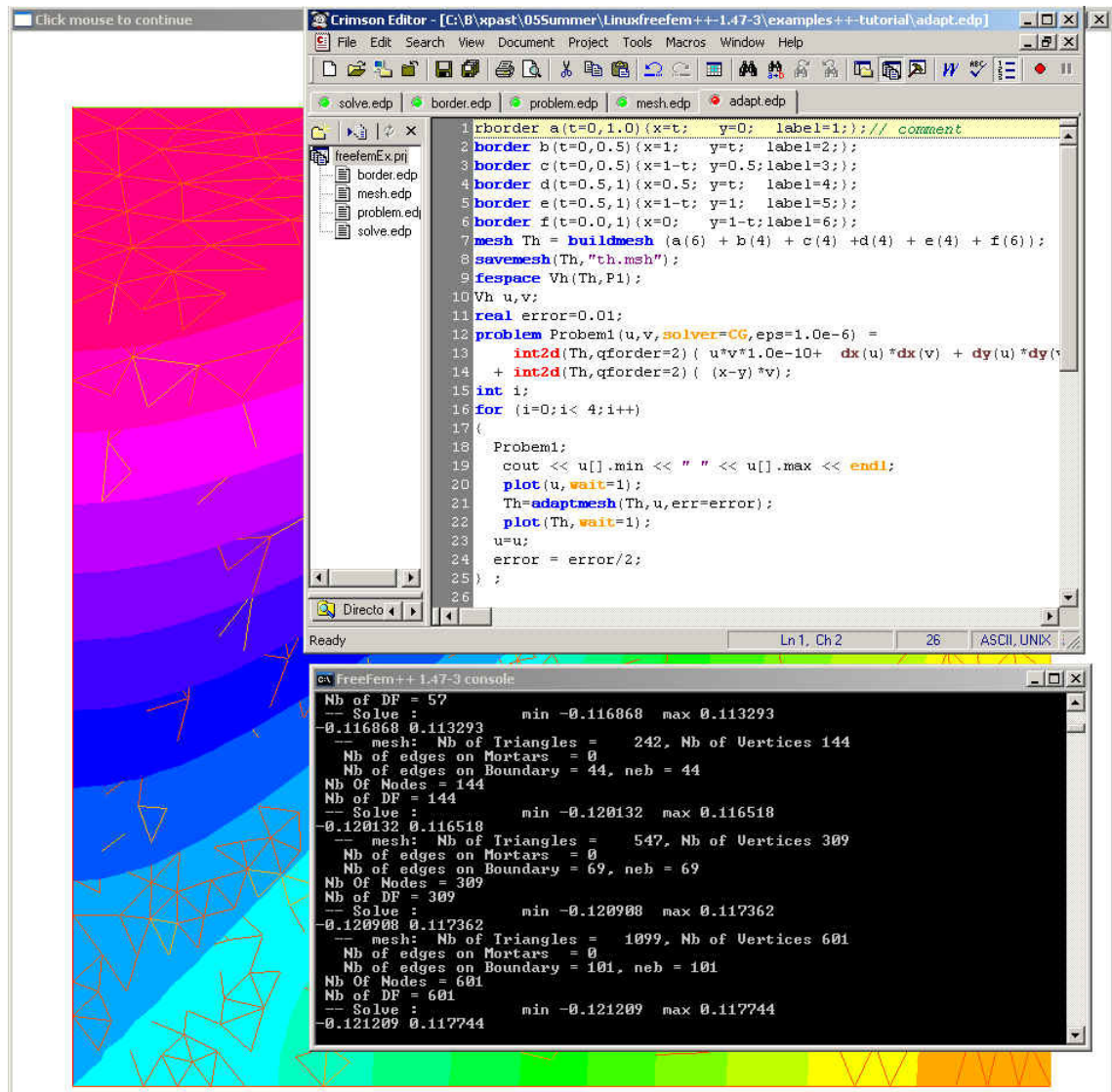


Figure 1.1: Integrated environment for FreeFem++ development with Windows

1.2 How to use FreeFem++

Under MacOS X with Graphic Interfaces To test an `.edp` file, just drag and drop the file icon on the MacOS application icon `FreeFem++.app`. You can also launch this application and use the menu: **File** → **Open**.

One of the best ways however on the Mac is to use a text editor like `Smultron.app` (see above).

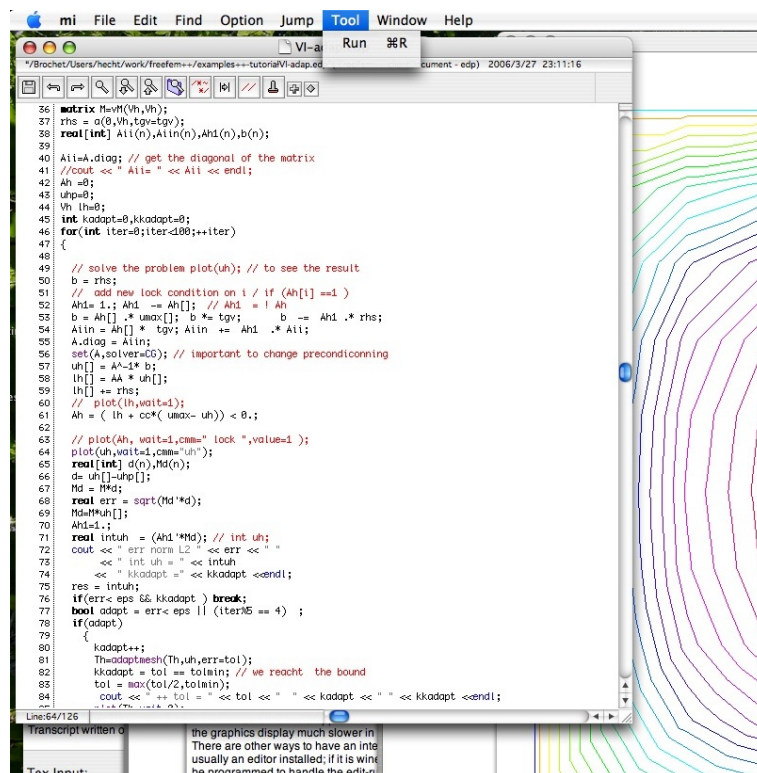


Figure 1.2: The 3 panels of the integrated environment built with the fraise Editor with FreeFem++ in action. The Tools menu has an item to launch FreeFem++ by a `Ctrl+1` command.

In Terminal mode Choose the type of application from `FreeFem++`, `FreeFem++-nw`, `FreeFem++-mpi`, ... according to your needs. Add at least the path name; for example

```
FreeFem++ your-edp-file-path
```

1.3 Environment variables, and the init file

FreeFem++ reads a user's init file named `freefem++.pref` to initialize global variables: `verbosity`, `includepath`, `loadpath`.

Remark 3 The variable `verbosity` changes the level of internal printing (0, nothing (unless there are syntax errors), 1 few, 10 lots, etc. ...), the default value is 2.

The include files are searched from the `includepath` list and the load files are searched from `loadpath` list.

The syntax of the file is:

```

verbosity= 5
loadpath += "/Library/FreeFem++/lib"
loadpath += "/Users/hecht/Library/FreeFem++/lib"
includepath += "/Library/FreeFem++/edp"
includepath += "/Users/hecht/Library/FreeFem++/edp"
# comment
load += "funcTemplate"
load += "myfunction"
load += "MUMPS_seq"

```

The possible paths for this file are

- under unix and MacOS

```

/etc/freefem++.pref
$(HOME)/.freefem++.pref
freefem++.pref

```

- under windows

```

freefem++.pref

```

We can also use shell environment variable to change verbosity and the search rule before the init files.

```

export FF_VERBOSITY=50
export FF_INCLUDEPATH="dir;;dir2"
export FF_LOADPATH="dir;;dir3"

```

Remark: the separator between directories must be ";;" and not ":" because ":" is used under Windows.

Remark, to show the list of init of FreeFem++ , do

```

export FF_VERBOSITY=100; ./FreeFem++-nw
-- verbosity is set to 100
insert init-files /etc/freefem++.pref $
...

```

1.4 History

The project has evolved from MacFem, PCfem, written in Pascal. The first C version lead to freefem 3.4; it offered mesh adaptativity on a single mesh only.

A thorough rewriting in C++ led to freefem+ (freefem+ 1.2.10 was its last release), which included interpolation over multiple meshes (functions defined on one mesh can be used on any other mesh); this software is no longer maintained but still in use because it

handles a problem description using the strong form of the PDEs. Implementing the interpolation from one unstructured mesh to another was not easy because it had to be fast and non-diffusive; for each point, one had to find the containing triangle. This is one of the basic problems of computational geometry (see Preparata & Shamos[18] for example). Doing it in a minimum number of operations was the challenge. Our implementation is $O(n \log n)$ and based on a quadtree. This version also grew out of hand because of the evolution of the template syntax in C++.

We have been working for a few years now on `FreeFem++` , entirely re-written again in C++ with a thorough usage of `template` and generic programming for coupled systems of unknown size at compile time. Like all versions of `freefem` it has a high level user friendly input language which is not too far from the mathematical writing of the problems.

The `freefem` language allows for a quick specification of any partial differential system of equations. The language syntax of `FreeFem++` is the result of a new design which makes use of the STL [26], templates and `bison` for its implementation; more detail can be found in [12]. The outcome is a versatile software in which any new finite element can be included in a few hours; but a recompilation is then necessary. Therefore the library of finite elements available in `FreeFem++` will grow with the version number and with the number of users who program more new elements. So far we have discontinuous P_0 elements, linear P_1 and quadratic P_2 Lagrangian elements, discontinuous P_1 and Raviart-Thomas elements and a few others like bubble elements.

Chapter 2

Getting Started

To illustrate with an example, let us explain how FreeFem++ solves **Poisson's** equation: *for a given function $f(x, y)$, find a function $u(x, y)$ satisfying*

$$-\Delta u(x, y) = f(x, y) \quad \text{for all } (x, y) \in \Omega, \quad (2.1)$$

$$u(x, y) = 0 \quad \text{for all } (x, y) \text{ on } \partial\Omega, . \quad (2.2)$$

Here $\partial\Omega$ is the boundary of the bounded open set $\Omega \subset \mathbb{R}^2$ and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

The following is a FreeFem++ program which computes u when $f(x, y) = xy$ and Ω is the unit disk. The boundary $C = \partial\Omega$ is

$$C = \{(x, y) \mid x = \cos(t), y = \sin(t), 0 \leq t \leq 2\pi\}$$

Note that in FreeFem++ the domain Ω is assumed to be described by its boundary that is on the left side of its boundary oriented by the parameter. As illustrated in Fig. 2.2, we can see the isovalue of u by using `plot` (see line 13 below).

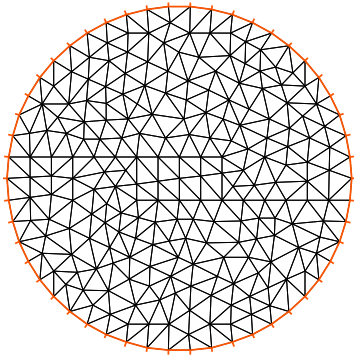


Figure 2.1: mesh Th by `build(C(50))`

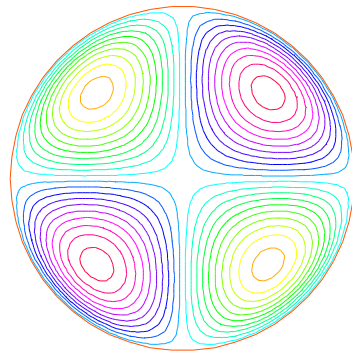


Figure 2.2: isovalue by `plot(u)`

Example 2.1

```
// defining the boundary
1: border C(t=0,2*pi){x=cos(t); y=sin(t);}
```

```

// the triangulated domain Th is on the left side of its boundary
2: mesh Th = buildmesh (C(50));
// the finite element space defined over Th is called here Vh
3: fespace Vh(Th,P1);
4: Vh u,v; // defines u and v as piecewise-P1 continuous functions
5: func f= x*y; // definition of a called f function
6: real cpu=clock(); // get the clock in second
7: solve Poisson(u,v,solver=LU) = // defines the PDE
8:   int2d(Th) (dx(u)*dx(v) + dy(u)*dy(v)) // bilinear part
9:   - int2d(Th) ( f*v) // right hand side
10:   + on(C,u=0) ; // Dirichlet boundary condition
11: plot(u);
12: cout << " CPU time = " << clock()-cpu << endl;

```

Note that the qualifier `solver=LU` is not required and by default a multi-frontal LU would have been used. Note also that the lines containing `clock` are equally not required. Finally note how close to the mathematics `FreeFem++` input language is. Line 8 and 9 correspond to the mathematical variational equation

$$\int_{T_h} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy = \int_{T_h} f v dx dy$$

for all v which are in the finite element space V_h and zero on the boundary C .

Exercise : Change P1 into P2 and run the program.

2.0.1 FEM by `FreeFem++` : how does it work?

This first example shows how `FreeFem++` executes with no effort all the usual steps required by the finite element method (FEM). Let us go through them one by one.

1st line: the boundary Γ is described analytically by a parametric equation for x and for y . When $\Gamma = \sum_{j=0}^J \Gamma_j$ then each curve Γ_j , must be specified and crossings of Γ_j are not allowed except at end points .

The keyword “label” can be added to define a group of boundaries for later use (boundary conditions for instance). Hence the circle could also have been described as two half circle with the same label:

```

border Gamma1(t=0,pi) {x=cos(t); y=sin(t); label=C}
border Gamma2(t=pi,2*pi) {x=cos(t); y=sin(t); label=C}

```

Boundaries can be referred to either by name (`Gamma1` for example) or by label (`C` here) or even by its internal number here 1 for the first half circle and 2 for the second (more examples are in Section 5.8).

2nd line: the triangulation \mathcal{T}_h of Ω is automatically generated by `buildmesh(C(50))` using 50 points on C as in Fig. 2.1.

The domain is assumed to be on the left side of the boundary which is implicitly oriented by the parametrization. So an elliptic hole can be added by

```
border C(t=2*pi,0){x=0.1+0.3*cos(t); y=0.5*sin(t);}
```

If by mistake one had written

```
border C(t=0,2*pi){x=0.1+0.3*cos(t); y=0.5*sin(t);}
```

then the inside of the ellipse would be triangulated as well as the outside.

Automatic mesh generation is based on the Delaunay-Voronoi algorithm. Refinement of the mesh are done by increasing the number of points on Γ , for example, `buildmesh(C(100))`, because inner vertices are determined by the density of points on the boundary. Mesh adaptation can be performed also against a given function f by calling `adaptmesh(Th,f)`. Now the name \mathcal{T}_h (Th in `FreeFem++`) refers to the family $\{T_k\}_{k=1,\dots,n_t}$ of triangles shown in figure 2.1. Traditionally h refers to the mesh size, n_t to the number of triangles in \mathcal{T}_h and n_v to the number of vertices, but it is seldom that we will have to use them explicitly. If Ω is not a polygonal domain, a “skin” remains between the exact domain Ω and its approximation $\Omega_h = \cup_{k=1}^{n_t} T_k$. However, we notice that all corners of $\Gamma_h = \partial\Omega_h$ are on Γ .

3rd line: A finite element space is, usually, a space of polynomial functions on elements, triangles here only, with certain matching properties at edges, vertices etc. Here `fespace Vh(Th,P1)` defines V_h to be the space of continuous functions which are affine in x, y on each triangle of T_h . As it is a linear vector space of finite dimension, basis can be found. The canonical basis is made of functions, called the *hat functions* ϕ_k which are continuous piecewise affine and are equal to 1 on one vertex and 0 on all others. A typical hat function is shown on figure 2.4¹. Then

$$V_h(\mathcal{T}_h, P_1) = \left\{ w(x, y) \left| w(x, y) = \sum_{k=1}^M w_k \phi_k(x, y), w_k \text{ are real numbers} \right. \right\} \quad (2.3)$$

where M is the dimension of V_h , i.e. the number of vertices. The w_k are called the *degree of freedom* of w and M the number of the degree of freedom.

It is said also that the *nodes* of this finite element method are the vertices.

Currently `FreeFem++` implements the following elements in 2d, (see section 6 for the full description)

P0 piecewise constant,

P1 continuous piecewise linear,

P2 continuous piecewise quadratic,

P3 continuous piecewise cubic, (need `load "Element_P3"`)

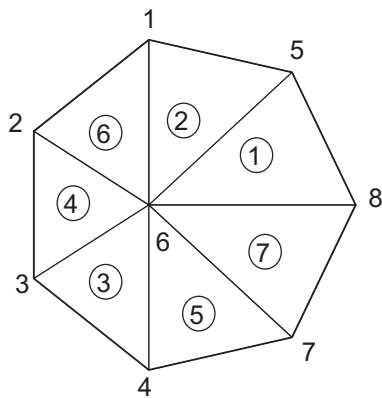
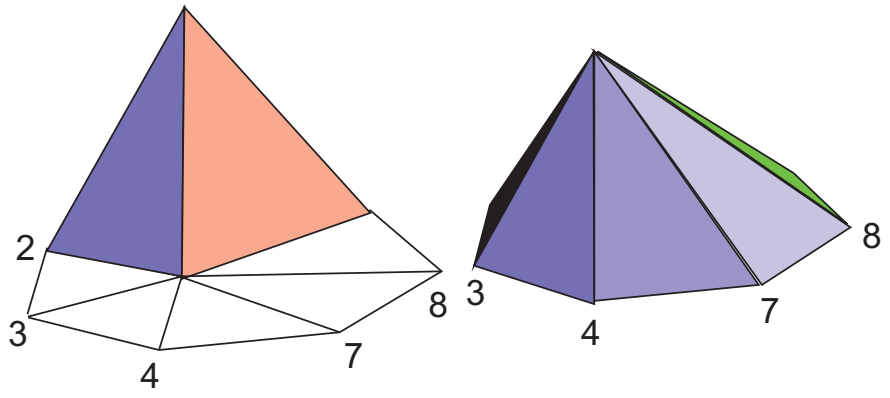
P4 continuous piecewise quartic, (need `load "Element_P4"`)

RT0 Raviart-Thomas piecewise constant,

¹ The easiest way to define ϕ_k is by making use of the *barycentric coordinates* $\lambda_i(x, y)$, $i = 1, 2, 3$ of a point $q = (x, y) \in T$, defined by

$$\sum_i \lambda_i = 1, \quad \sum_i \lambda_i q^i = q$$

where q^i , $i = 1, 2, 3$ are the 3 vertices of T . Then it is easy to see that the restriction of ϕ_k on T is precisely λ_k .

Figure 2.3: mesh \mathcal{T}_h Figure 2.4: Graph of ϕ_1 (left) and ϕ_6

RT1 Raviart-Thomas degree 1 piecewise constant (need `load "Element_Mixte"`)
 BDM1 Brezzi-Douglas-Marini degree 1 piecewise constant (need `load "Element_Mixte"`)
 RT0Ortho Nedelec type 1 degree 0 piecewise constant
 RT1Ortho Nedelec type 1 degree 1 piecewise constant (need `load "Element_Mixte"`)
 BDM1Ortho Brezzi-Douglas-Marini degree 1 piecewise constant (need `load "Element_Mixte"`)
 P1nc piecewise linear non-conforming,
 P1dc piecewise linear discontinuous,
 P2dc piecewise quadratic discontinuous,
 P2h quadratic homogeneous continuous (without P1)
 P3dc piecewise cubic discontinuous,(need `load "Element_P3dc"`)
 P4dc piecewise quartic discontinuous,(need `load "Element_P4dc"`)
 P1b piecewise linear continuous plus bubble,
 P2b piecewise quadratic continuous plus bubble.
 Morley Morley finite element (need `load "Morley"`)
 P2BR P2 Bernardi-Raugel finite element (need `load "BernardiRaugel.cpp"`)
 P0edge a finite element constant per edge
 P1edge to P5edge a finite element polynomial on edge (need `load "Element_PkEdge"`)
 ...
 Currently FreeFem++ implements the following elements in 3d, (see section 6 for the full description)
 P03d piecewise constant,
 P13d continuous piecewise linear,
 P23d continuous piecewise quadratic,
 RT03d Raviart-Thomas piecewise constant,
 Edge03d The Nedelec Edge element
 P1b3d piecewise linear continuous plus bubble,
 ...

To get the full list, in a unix terminal, in directory `examples++-tutorial` do

```
FreeFem++ dumptable.edp
grep TypeOfFE lestables
```

Note that other elements can be added fairly easily.

Step3: Setting the problem

4th line: `Vh u, v` declares that u and v are approximated as above, namely

$$u(x, y) \simeq u_h(x, y) = \sum_{k=0}^{M-1} u_k \phi_k(x, y) \quad (2.4)$$

5th line: the right hand side f is defined analytically using the keyword `func`.

7th–9th lines: defines the bilinear form of equation (2.1) and its Dirichlet boundary conditions (2.2).

This *variational formulation* is derived by multiplying (2.1) by $v(x, y)$ and integrating the result over Ω :

$$-\int_{\Omega} v \Delta u \, dx dy = \int_{\Omega} v f \, dx dy$$

Then, by Green's formula, the problem is converted into finding u such that

$$a(u, v) - \ell(f, v) = 0 \quad \forall v \text{ satisfying } v = 0 \text{ on } \partial\Omega. \quad (2.5)$$

$$\text{with } a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx dy, \quad \ell(f, v) = \int_{\Omega} f v \, dx dy \quad (2.6)$$

In FreeFem++ the **Poisson** problem can be declared only as in

```
Vh u, v; problem Poisson(u, v) =
and solved later as in

...
Poisson;                                // the problem is solved here
...
```

or declared and solved at the same time as in

```
Vh u, v; solve Poisson(u, v) = int(...
```

and (2.5) is written with `dx(u) = $\partial u / \partial x$` , `dy(u) = $\partial u / \partial y$` and

$$\begin{aligned} \int_{\Omega} \nabla u \cdot \nabla v \, dx dy &\longrightarrow \text{int2d}(\text{Th}) (\text{dx}(u) * \text{dx}(v) + \text{dy}(u) * \text{dy}(v)) \\ \int_{\Omega} f v \, dx dy &\longrightarrow \text{int2d}(\text{Th}) (f * v) \quad (\text{Notice here, } u \text{ is unused}) \end{aligned}$$

In FreeFem++ **bilinear terms and linear terms should not be under the same integral**; indeed to construct the linear systems FreeFem++ finds out which integral contributes to the bilinear form by checking if both terms, the unknown (here u) and test functions (here v) are present.

Step4: Solution and visualization

6th line: The current time in seconds is stored into the real-valued variable `cpu`.

7th line The problem is solved.

11th line: The visualization is done as illustrated in Fig. 2.2 (see Section 7.1 for zoom, postscript and other commands).

12th line: The computing time (not counting graphics) is written on the console Notice the C++-like syntax; the user needs not study C++ for using FreeFem++ , but it helps to guess what is allowed in the language.

Access to matrices and vectors

Internally FreeFem++ will solve a linear system of the type

$$\sum_{j=0}^{M-1} A_{ij} u_j - F_i = 0, \quad i = 0, \dots, M-1; \quad F_i = \int_{\Omega} f \phi_i \, dx dy \quad (2.7)$$

which is found by using (2.4) and replacing v by ϕ_i in (2.5). And the Dirichlet conditions are implemented by penalty, namely by setting $A_{ii} = 10^{30}$ and $F_i = 10^{30} * 0$ if i is a boundary degree of freedom. Note, that the number 10^{30} is called `tg` (*très grande valeur*) and it is generally possible to change this value , see the index item `solve!tg`.

The matrix $A = (A_{ij})$ is called *stiffness matrix* .

If the user wants to access A directly he can do so by using (see section 6.12 page 186 for details)

```
varf a(u,v) = int2d(Th) ( dx(u)*dx(v) + dy(u)*dy(v) )
               + on(C,u=0) ;
matrix A=a(Vh,Vh); // stiffness matrix,
```

The vector F in (2.7) can also be constructed manually

```
varf l(unused,v) = int2d(Th) (f*v)+on(C,unused=0);
Vh F; F[] = l(0,Vh); // F[] is the vector associated to the function F
```

The problem can then be solved by

```
u[] = A^-1 * F[]; // u[] is the vector associated to the function u
```

Note 2.1 Here u and F are finite element function, and $u[]$ and $F[]$ give the array of value associated ($u[] \equiv (u_i)_{i=0,\dots,M-1}$ and $F[] \equiv (F_i)_{i=0,\dots,M-1}$). So we have

$$u(x, y) = \sum_{i=0}^{M-1} u[i] \phi_i(x, y), \quad F(x, y) = \sum_{i=0}^{M-1} F[i] \phi_i(x, y)$$

where $\phi_i, i = 0, \dots, M-1$ are the basis functions of V_h like in equation (2.3), and $M = V_h.\text{ndof}$ is the number of degree of freedom (i.e. the dimension of the space V_h).

The linear system (2.7) is solved by UMFPACK unless another option is mentioned specifically as in

```
Vh u,v; problem Poisson(u,v,solver=CG) = int2d(...
```

meaning that `Poisson` is declared only here and when it is called (by simply writing `Poisson;`) then (2.7) will be solved by the Conjugate Gradient method.

2.0.2 Some Features of FreeFem++

The language of FreeFem++ is typed, polymorphic and reentrant with macro generation (see 9.12). Every variable must be typed and declared in a statement each statement separated from the next by a semicolon ";". The syntax is that of C++ by default augmented with something that is more akin to T_EX. For the specialist, one key guideline is that FreeFem++ rarely generates an internal finite element array; this was adopted for speed and consequently FreeFem++ could be hard to beat in terms of execution speed, except for the time lost in the interpretation of the language (which can be reduced by a systematic usage of `varf` and matrices instead of `problem`).

2.1 The Development Cycle: Edit–Run/Visualize–Revise

An integrated environment is provided with FreeFem++ by A. Le Hyaric; Many examples and tutorials are also given along with this documentation and it is best to study them and learn by example. Explanations for some of these examples are given in this documentation in the next chapter. If you are a FEM beginner, you may also have to read a book on variational formulations.

The development cycle will have the following steps:

Modeling: From strong forms of PDE to weak forms, one must know the variational formulation to use FreeFem++ ; one should also have an eye on the reusability of the variational formulation so as to keep the same internal matrices; a typical example is the time dependent heat equation with an implicit time scheme: the internal matrix can be factorized only once and FreeFem++ can be taught to do so.

Programming: Write the code in FreeFem++ language using a text editor such as the one provided in the integrated environment.

Run: Run the code (here written in file `mycode.edp`). note that this can also be done in terminal mode by :

```
% FreeFem++ mycode.edp
```

Visualization: Use the keyword `plot` to display functions while FreeFem++ is running. Use the plot-parameter `wait=1` to stop the program at each plot. Use the plot-parameter `ps="toto.eps"` to generate a postscript file to archive the results.

Debugging: A global variable "debug" (for example) can help as in `wait=true` to `wait=false`.

```
bool debug = true;
border a(t=0,2*pi){ x=cos(t); y=sin(t);label=1;}
border b(t=0,2*pi){ x=0.8+0.3*cos(t); y=0.3*sin(t);label=2;}
plot(a(50)+b(-30),wait=debug); // plot the borders to see the intersection
// (so change (0.8 in 0.3 in b) then needs a mouse click
mesh Th = buildmesh(a(50)+b(-30));
plot(Th,wait=debug); // plot Th then needs a mouse click
fespace Vh(Th,P2);
Vh f = sin(pi*x)*cos(pi*y);
plot(f,wait=debug); // plot the function f
```

```
Vh g = sin(pi*x + cos(pi*y));
plot(g,wait=debug); // plot the function g
```

Changing debug to false will make the plots flow continuously; watching the flow of graphs on the screen (while drinking coffee) can then become a pleasant experience.

Error messages are displayed in the console window. They are not always very explicit because of the template structure of the C++ code, (we did our best)! Nevertheless they are displayed at the right place. For example, if you forget parenthesis as in

```
bool debug = true;
mesh Th = square(10,10;
plot(Th);
```

then you will get the following message from FreeFem++ ,

```
2 : mesh Th = square(10,10;
Error line number 2, in file bb.edp, before token ;
parse error
current line = 2
Compile error : parse error
line number :2, ;
error Compile error : parse error
line number :2, ;
code = 1
```

If you use the same symbol twice as in

```
real aaa =1;
real aaa;
```

then you will get the message

```
2 : real aaa; The identifier aaa exists
the existing type is <Pd>
the new type is <Pd>
```

If you find that the program isn't doing what you want you may also use `cout` to display in text format on the console window the value of variables, just as you would do in C++.

The following example works:

```
...;
fespace Vh...; Vh u;...
cout<<u;...
matrix A=a(Vh,Vh);...
cout<<A;
```

Another trick is to *comment in and out* by using the “//” as in C++. For example

```
real aaa =1;
// real aaa;
```

Chapter 3

Learning by Examples

This chapter is for those, like us, who don't like to read manuals. A number of simple examples cover a good deal of the capacity of FreeFem++ and are self-explanatory. For the modeling part this chapter continues at Chapter 9 where some PDEs of physics, engineering and finance are studied in greater depth.

3.1 Membranes

Summary *Here we shall learn how to solve a Dirichlet and/or mixed Dirichlet Neumann problem for the Laplace operator with application to the equilibrium of a membrane under load. We shall also check the accuracy of the method and interface with other graphics packages.*

An elastic membrane Ω is attached to a planar rigid support Γ , and a force $f(x)dx$ is exerted on each surface element $dx = dx_1dx_2$. The vertical membrane displacement, $\varphi(x)$, is obtained by solving Laplace's equation:

$$-\Delta\varphi = f \text{ in } \Omega.$$

As the membrane is fixed to its planar support, one has:

$$\varphi|_{\Gamma} = 0.$$

If the support wasn't planar but at an elevation $z(x_1, x_2)$ then the boundary conditions would be of non-homogeneous Dirichlet type.

$$\varphi|_{\Gamma} = z.$$

If a part Γ_2 of the membrane border Γ is not fixed to the support but is left hanging, then due to the membrane's rigidity the angle with the normal vector n is zero; thus the boundary conditions are

$$\varphi|_{\Gamma_1} = z, \quad \frac{\partial\varphi}{\partial n}|_{\Gamma_2} = 0$$

where $\Gamma_1 = \Gamma - \Gamma_2$; recall that $\frac{\partial\varphi}{\partial n} = \nabla\varphi \cdot n$. Let us recall also that the Laplace operator Δ is defined by:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2}.$$

With such "mixed boundary conditions" the problem has a unique solution (see (1987), Dautray-Lions (1988), Strang (1986) and Raviart-Thomas (1983)); the easiest proof is to notice that φ is the state of least energy, i.e.

$$E(\phi) = \min_{\varphi-z \in V} E(v), \quad \text{with} \quad E(v) = \int_{\Omega} \left(\frac{1}{2} |\nabla v|^2 - fv \right)$$

and where V is the subspace of the Sobolev space $H^1(\Omega)$ of functions which have zero trace on Γ_1 . Recall that ($x \in \mathbb{R}^d$, $d = 2$ here)

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in (L^2(\Omega))^d\}$$

Calculus of variation shows that the minimum must satisfy, what is known as the weak form of the PDE or its variational formulation (also known here as the theorem of virtual work)

$$\int_{\Omega} \nabla \varphi \cdot \nabla w = \int_{\Omega} fw \quad \forall w \in V$$

Next an integration by parts (Green's formula) will show that this is equivalent to the PDE when second derivatives exist.

WARNING Unlike `freefem+` which had both weak and strong forms, `FreeFem++` implements only weak formulations. It is not possible to go further in using this software if you don't know the weak form (i.e. variational formulation) of your problem: either you read a book, or ask help from a colleague or drop the matter. Now if you want to solve a system of PDE like $A(u, v) = 0$, $B(u, v) = 0$ don't close this manual, because in weak form it is

$$\int_{\Omega} (A(u, v)w_1 + B(u, v)w_2) = 0 \quad \forall w_1, w_2 \dots$$

Example Let an ellipse have the length of the semimajor axis $a = 2$, and unitary the semiminor axis Let the surface force be $f = 1$. Programming this case with `FreeFem++` gives:

```

Example 3.1 (membrane.edp)                                     //    file membrane.edp
real theta=4.*pi/3.;
real a=2.,b=1.;           //    the length of the semimajor axis and semiminor axis
func z=x;

border Gamma1(t=0,theta)    { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2*pi) { x = a * cos(t); y = b*sin(t); }
mesh Th=buildmesh(Gamma1(100)+Gamma2(50));

fespace Vh(Th,P2);           //    P2 conforming triangular FEM
Vh phi,w, f=1;

solve Laplace(phi,w)=int2d(Th) (dx(phi)*dx(w) + dy(phi)*dy(w))
                        - int2d(Th) (f*w) + on(Gamma1,phi=z);
plot(phi,wait=true, ps="membrane.eps");           //    Plot phi
plot(Th,wait=true, ps="membraneTh.eps");          //    Plot Th

savemesh(Th,"Th.msh");
```

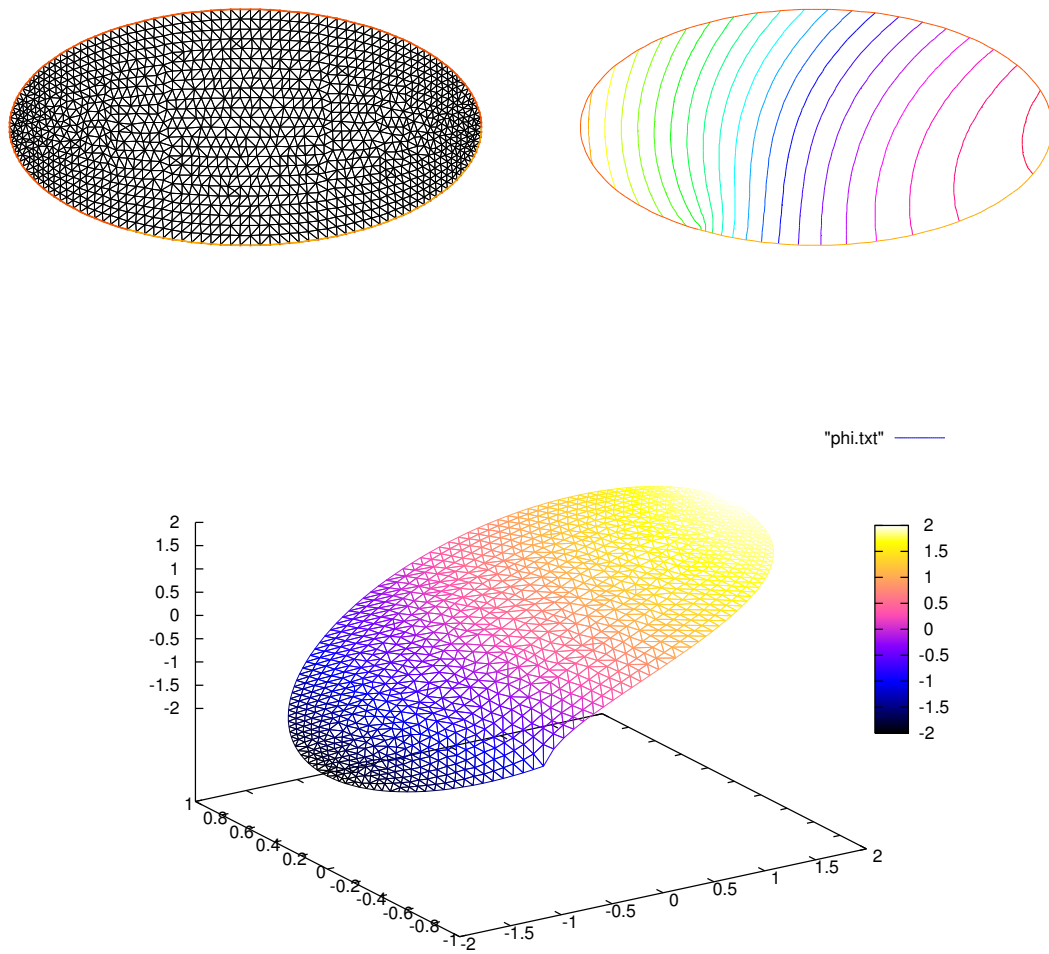


Figure 3.1: Mesh and level lines of the membrane deformation. Below: the same in 3D drawn by gnuplot from a file generated by FreeFem++ .

A triangulation is built by the keyword `buildmesh`. This keyword calls a triangulation subroutine based on the Delaunay test, which first triangulates with only the boundary points, then adds internal points by subdividing the edges. How fine the triangulation becomes is controlled by the size of the closest boundary edges.

The PDE is then discretized using the triangular second order finite element method on the triangulation; as was briefly indicated in the previous chapter, a linear system is derived from the discrete formulation whose size is the number of vertices plus the number of mid-edges in the triangulation. The system is solved by a multi-frontal Gauss LU factorization implemented in the package UMFPACK. The keyword `plot` will display both T_h and φ (remove `Th` if φ only is desired) and the qualifier `fill=true` replaces the default option (colored level lines) by a full color display. Results are on figure 3.1.

```
plot(phi,wait=true,fill=true);           // Plot phi with full color display
```

Next we would like to check the results!

One simple way is to adjust the parameters so as to know the solutions. For instance on the unit circle $a=1$, $\varphi_e = \sin(x^2 + y^2 - 1)$ solves the problem when

$$z = 0, \quad f = -4(\cos(x^2 + y^2 - 1) - (x^2 + y^2) \sin(x^2 + y^2 - 1))$$

except that on Γ_2 $\partial_n \varphi = 2$ instead of zero. So we will consider a non-homogeneous Neumann condition and solve

$$\int_{\Omega} (\nabla \varphi \cdot \nabla w) = \int_{\Omega} f w + \int_{\Gamma_2} 2w \quad \forall w \in V$$

We will do that with two triangulations, compute the L^2 error:

$$\epsilon = \int_{\Omega} |\varphi - \varphi_e|^2$$

and print the error in both cases as well as the log of their ratio an indication of the rate of convergence.

Example 3.2 (membranerror.edp)

```
// file membranerror.edp
verbosity=0; // to remove all default output
real theta=4.*pi/3.;
real a=1.,b=1.; // the length of the semimajor axis and semiminor axis
border Gamma1(t=0,theta) { x = a * cos(t); y = b*sin(t); }
border Gamma2(t=theta,2*pi) { x = a * cos(t); y = b*sin(t); }

func f=-4*(cos(x^2+y^2-1) - (x^2+y^2)*sin(x^2+y^2-1));
func phiexact=sin(x^2+y^2-1);

real[int] L2error(2); // an array two values
for(int n=0;n<2;n++)
{
  mesh Th=buildmesh(Gamma1(20*(n+1))+Gamma2(10*(n+1)));
  fespace Vh(Th,P2);
  Vh phi,w;

  solve laplace(phi,w)=int2d(Th)(dx(phi)*dx(w) + dy(phi)*dy(w))
    - int2d(Th)(f*w) - int1d(Th,Gamma2)(2*w)+ on(Gamma1,phi=0);
```

```

    plot(Th, phi, wait=true, ps="membrane.eps");           // Plot Th and phi

    L2error[n]= sqrt(int2d(Th)((phi-phiexact)^2));
}

for(int n=0;n<2;n++)
    cout << " L2error " << n << " = "<< L2error[n] <<endl;

cout <<" convergence rate = "<< log(L2error[0]/L2error[1])/log(2.) <<endl;

```

the output is

```

L2error 0 = 0.00462991
L2error 1 = 0.00117128
convergence rate = 1.9829
times: compile 0.02s, execution 6.94s

```

We find a rate of 1.93591, which is not close enough to the 3 predicted by the theory. The Geometry is always a polygon so we lose one order due to the geometry approximation in $O(h^2)$

Now if you are not satisfied with the .eps plot generated by FreeFem++ and you want to use other graphic facilities, then you must store the solution in a file very much like in C++. It will be useless if you don't save the triangulation as well, consequently you must do

```

{
    ofstream ff("phi.txt");
    ff << phi[];
}
savemesh(Th, "Th.msh");

```

For the triangulation the name is important: it is the extension that determines the format.

Still that may not take you where you want. Here is an interface with gnuplot to produce the right part of figure 3.2.

```

// to build a gnuplot data file
{ ofstream ff("graph.txt");
  for (int i=0;i<Th.nt;i++)
  { for (int j=0; j <3; j++)
    ff<<Th[i][j].x << " " << Th[i][j].y<< " " <<phi[][Vh(i,j)]<<endl;
    ff<<Th[i][0].x << " " << Th[i][0].y<< " " <<phi[][Vh(i,0)]<<"\n\n\n"
  }
}

```

We use the finite element numbering, where $Vh(i, j)$ is the global index of j^{th} degrees of freedom of triangle number i .

Then open gnuplot and do

```

set palette rgbformulae 30,31,32
splot "graph.txt" w l pal

```

This works with P2 and P1, but not with P1nc because the 3 first degrees of freedom of P2 or P2 are on vertices and not with P1nc.

3.2 Heat Exchanger

Summary Here we shall learn more about geometry input and triangulation files, as well as read and write operations.

The problem Let $\{C_i\}_{1,2}$, be 2 thermal conductors within an enclosure C_0 . The first one is held at a constant temperature u_1 the other one has a given thermal conductivity κ_2 5 times larger than the one of C_0 . We assume that the border of enclosure C_0 is held at temperature $20^\circ C$ and that we have waited long enough for thermal equilibrium.

In order to know $u(x)$ at any point x of the domain Ω , we must solve

$$\nabla \cdot (\kappa \nabla u) = 0 \quad \text{in } \Omega, \quad u|_\Gamma = g$$

where Ω is the interior of C_0 minus the conductors C_1 and Γ is the boundary of Ω , that is $C_0 \cup C_1$. Here g is any function of x equal to u_i on C_i . The second equation is a reduced form for:

$$u = u_i \quad \text{on } C_i, \quad i = 0, 1.$$

The variational formulation for this problem is in the subspace $H_0^1(\Omega) \subset H^1(\Omega)$ of functions which have zero traces on Γ .

$$u - g \in H_0^1(\Omega) : \int_{\Omega} \nabla u \nabla v = 0 \quad \forall v \in H_0^1(\Omega)$$

Let us assume that C_0 is a circle of radius 5 centered at the origin, C_i are rectangles, C_1 being at the constant temperature $u_1 = 60^\circ C$.

Example 3.3 (heatex.edp)

```
// file heatex.edp
int C1=99, C2=98; // could be anything such that ≠ 0 and C1 ≠ C2
border C0(t=0,2*pi){x=5*cos(t); y=5*sin(t);}

border C11(t=0,1){ x=1+t; y=3; label=C1;}
border C12(t=0,1){ x=2; y=3-6*t; label=C1;}
border C13(t=0,1){ x=2-t; y=-3; label=C1;}
border C14(t=0,1){ x=1; y=-3+6*t; label=C1;}

border C21(t=0,1){ x=-2+t; y=3; label=C2;}
border C22(t=0,1){ x=-1; y=3-6*t; label=C2;}
border C23(t=0,1){ x=-1-t; y=-3; label=C2;}
border C24(t=0,1){ x=-2; y=-3+6*t; label=C2;}

plot( C0(50) // to see the border of the domain
+ C11(5)+C12(20)+C13(5)+C14(20)
+ C21(-5)+C22(-20)+C23(-5)+C24(-20),
wait=true, ps="heatexb.eps");

mesh Th=buildmesh( C0(50)
+ C11(5)+C12(20)+C13(5)+C14(20)
+ C21(-5)+C22(-20)+C23(-5)+C24(-20));

plot(Th,wait=1);
```



```

fespace Vh(Th,P1); Vh u,v;
Vh kappa=1+2*(x<-1)*(x>-2)*(y<3)*(y>-3);
solve a(u,v)= int2d(Th) (kappa*(dx(u)*dx(v)+dy(u)*dy(v)))
               +on(C0,u=20)+on(C1,u=60);
plot(u,wait=true, value=true, fill=true, ps="heatex.eps");

```

Note the following:

- C0 is oriented counterclockwise by t , while C1 is oriented clockwise and C2 is oriented counterclockwise. This is why C1 is viewed as a hole by buildmesh.
- C1 and C2 are built by joining pieces of straight lines. To group them in the same logical unit to input the boundary conditions in a readable way we assigned a label on the boundaries. As said earlier, borders have an internal number corresponding to their order in the program (check it by adding a `cout<<C22;` above). This is essential to understand how a mesh can be output to a file and re-read (see below).
- As usual the mesh density is controlled by the number of vertices assigned to each boundary. It is not possible to change the (uniform) distribution of vertices but a piece of boundary can always be cut in two or more parts, for instance C12 could be replaced by C121+C122:

```

//      border C12(t=0,1) x=2; y=3-6*t; label=C1;
border C121(t=0,0.7){ x=2;    y=3-6*t;  label=C1;}
border C122(t=0.7,1){ x=2;    y=3-6*t;  label=C1;}
... buildmesh(.../* C12(20) */ + C121(12)+C122(8)+...);

```

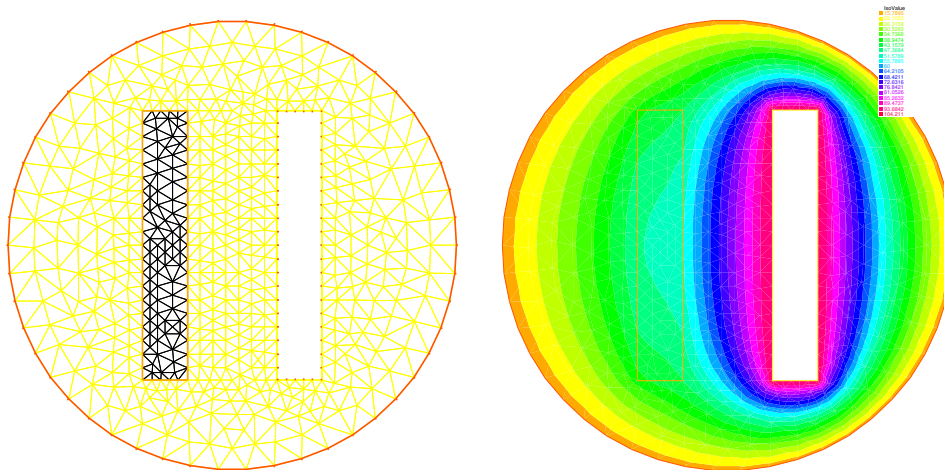


Figure 3.2: The heat exchanger

Exercise Use the symmetry of the problem with respect to the axes; triangulate only one half of the domain, and set Dirichlet conditions on the vertical axis, and Neumann conditions on the horizontal axis.

Writing and reading triangulation files Suppose that at the end of the previous program we added the line

```
savemesh(Th, "condensor.msh");
```

and then later on we write a similar program but we wish to read the mesh from that file. Then this is how the condenser should be computed:

```
mesh Sh=readmesh("condensor.msh");
fespace Wh(Sh,P1); Wh us,vs;
solve b(us,vs)= int2d(Sh) (dx(us)*dx(vs)+dy(us)*dy(vs))
+on(1,us=0)+on(99,us=1)+on(98,us=-1);
plot(us);
```

Note that the names of the boundaries are lost but either their internal number (in the case of C0) or their label number (for C1 and C2) are kept.

3.3 Acoustics

Summary *Here we go to grip with ill posed problems and eigenvalue problems*
Pressure variations in air at rest are governed by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0.$$

When the solution wave is monochromatic (and that depend on the boundary and initial conditions), u is of the form $u(x, t) = \text{Re}(v(x)e^{ikt})$ where v is a solution of Helmholtz's equation:

$$\begin{aligned} k^2 v + c^2 \Delta v &= 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial n} \Big|_{\Gamma} &= g. \end{aligned} \tag{3.1}$$

where g is the source. Note the “+” sign in front of the Laplace operator and that $k > 0$ is real. This sign may make the problem ill posed for some values of $\frac{c}{k}$, a phenomenon called “resonance”.

At resonance there are non-zero solutions even when $g = 0$. So the following program may or may not work:

Example 3.4 (sound.edp)

// file sound.edp

```
real kc2=1;
func g=y*(1-y);

border a0(t=0,1) { x= 5; y= 1+2*t ;}
border a1(t=0,1) { x=5-2*t; y= 3 ;}
border a2(t=0,1) { x= 3-2*t; y=3-2*t ;}
border a3(t=0,1) { x= 1-t; y= 1 ;}
border a4(t=0,1) { x= 0; y= 1-t ;}
border a5(t=0,1) { x= t; y= 0 ;}
border a6(t=0,1) { x= 1+4*t; y= t ;}
```

```

mesh Th=buildmesh( a0(20) + a1(20) + a2(20)
                    + a3(20) + a4(20) + a5(20) + a6(20));
fespace Vh(Th,P1);
Vh u,v;

solve sound(u,v)=int2d(Th)(u*v * kc2 - dx(u)*dx(v) - dy(u)*dy(v))
                - int1d(Th,a4)(g*v);
plot(u, wait=1, ps="sound.eps");

```

Results are on Figure 3.3. But when $kc2$ is an eigenvalue of the problem, then the solution is not unique: if $u_e \neq 0$ is an eigen state, then for any given solution $u + u_e$ is another a solution. To find all the u_e one can do the following

```

real sigma = 20;                                     // value of the shift
                                                    // OP = A - sigma B ; // the shifted matrix
varf op(u1,u2)= int2d(Th)( dx(u1)*dx(u2) + dy(u1)*dy(u2) - sigma* u1*u2 );
varf b([u1],[u2]) = int2d(Th)( u1*u2 ); // no Boundary condition see note
9.1

matrix OP= op(Vh,Vh,solver=Crout,factorize=1);
matrix B= b(Vh,Vh,solver=CG,eps=1e-20);

int nev=2;                                           // number of requested eigenvalues near sigma

real[int] ev(nev);                                 // to store the nev eigenvalue
Vh[int] eV(nev);                                   // to store the nev eigenvector

int k=EigenValue(OP,B,sym=true,sigma=sigma,value=ev,vector=eV,
                 tol=1e-10,maxit=0,ncv=0);
cout<<ev(0)<<" 2 eigen values "<<ev(1)<<endl;
v=eV[0];
plot(v,wait=1,ps="eigen.eps");

```

3.4 Thermal Conduction

Summary Here we shall learn how to deal with a time dependent parabolic problem. We shall also show how to treat an axisymmetric problem and show also how to deal with a nonlinear problem.

How air cools a plate We seek the temperature distribution in a plate $(0, Lx) \times (0, Ly) \times (0, Lz)$ of rectangular cross section $\Omega = (0, 6) \times (0, 1)$; the plate is surrounded by air at temperature u_e and initially at temperature $u = u_0 + \frac{x}{L}u_1$. In the plane perpendicular to the plate at $z = Lz/2$, the temperature varies little with the coordinate z ; as a first approximation the problem is 2D.

We must solve the temperature equation in Ω in a time interval $(0, T)$.

$$\begin{aligned}
 \partial_t u - \nabla \cdot (\kappa \nabla u) &= 0 \text{ in } \Omega \times (0, T), \\
 u(x, y, 0) &= u_0 + \frac{x}{L}u_1 \\
 \kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) &= 0 \text{ on } \Gamma \times (0, T).
 \end{aligned} \tag{3.2}$$

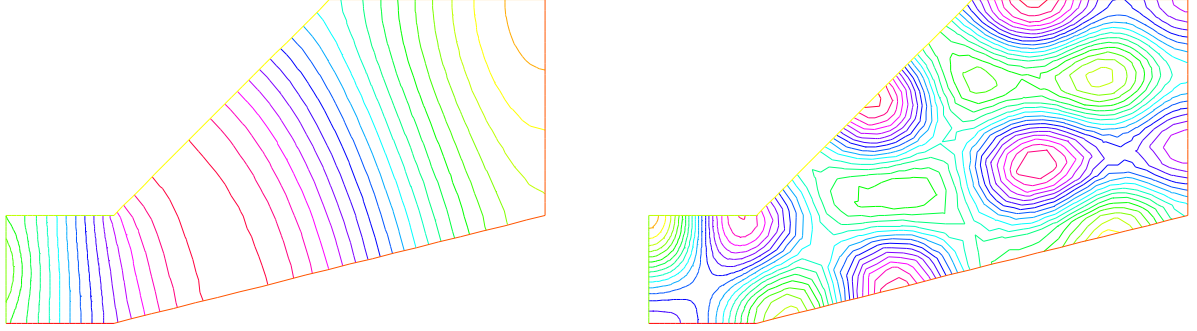


Figure 3.3: Left: Amplitude of an acoustic signal coming from the left vertical wall. Right: first eigen state ($\lambda = (k/c)^2 = 19.4256$) close to 20 of eigenvalue problem $-\Delta\varphi = \lambda\varphi$ and $\frac{\partial\varphi}{\partial n} = 0$ on Γ

Here the diffusion κ will take two values, one below the middle horizontal line and ten times less above, so as to simulate a thermostat. The term $\alpha(u - u_e)$ accounts for the loss of temperature by convection in air. Mathematically this boundary condition is of Fourier (or Robin, or mixed) type.

The variational formulation is in $L^2(0, T; H^1(\Omega))$; in loose terms and after applying an implicit Euler finite difference approximation in time; we shall seek $u^n(x, y)$ satisfying for all $w \in H^1(\Omega)$:

$$\int_{\Omega} \left(\frac{u^n - u^{n-1}}{\delta t} w + \kappa \nabla u^n \nabla w \right) + \int_{\Gamma} \alpha (u^n - u_e) w = 0$$

```

func u0 = 10 + 90 * x / 6;
func k = 1.8 * (y < 0.5) + 0.2;
real ue = 25, alpha = 0.25, T = 5, dt = 0.1 ;

mesh Th = square(30, 5, [6 * x, y]);
espace Vh(Th, P1);
Vh u = u0, v, uold;

problem thermic(u, v) = int2d(Th) (u * v / dt + k * (dx(u) * dx(v) + dy(u) * dy(v)))
    + int1d(Th, 1, 3) (alpha * u * v)
    - int1d(Th, 1, 3) (alpha * ue * v)
    - int2d(Th) (uold * v / dt) + on(2, 4, u = u0);

ofstream ff("thermic.dat");
for(real t = 0; t < T; t += dt) {
    uold = u; // uold ≡ un-1 = un ≡ u
    thermic; // here solve the thermic problem
    ff << u(3, 0.5) << endl;

```

```

    plot(u);
}

```

Notice that we must separate by hand the bilinear part from the linear one.

Notice also that the way we store the temperature at point (3,0.5) for all times in file `thermic.dat`. Should a one dimensional plot be required, the same procedure can be used. For instance to print $x \mapsto \frac{\partial u}{\partial y}(x, 0.9)$ one would do

```

for(int i=0;i<20;i++) cout<<dy(u)(6.0*i/20.0,0.9)<<endl;

```

Results are shown on Figure 3.4.

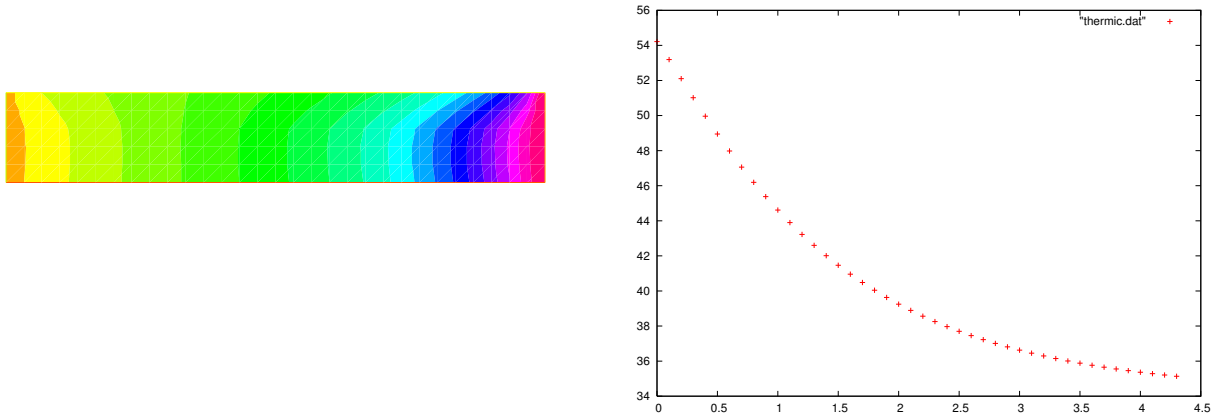


Figure 3.4: Temperature at T=4.9. Right: decay of temperature versus time at x=3, y=0.5

3.4.1 Axisymmetry: 3D Rod with circular section

Let us now deal with a cylindrical rod instead of a flat plate. For simplicity we take $\kappa = 1$. In cylindrical coordinates, the Laplace operator becomes (r is the distance to the axis, z is the distance along the axis, θ polar angle in a fixed plane perpendicular to the axis):

$$\Delta u = \frac{1}{r} \partial_r (r \partial_r u) + \frac{1}{r^2} \partial_{\theta\theta}^2 u + \partial_{zz}^2.$$

Symmetry implies that we loose the dependence with respect to θ ; so the domain Ω is again a rectangle $]0, R[\times]0, L[$. We take the convention of numbering of the edges as in `square()` (1 for the bottom horizontal ...); the problem is now:

$$\begin{aligned}
 r \partial_t u - \partial_r (r \partial_r u) - \partial_z (r \partial_z u) &= 0 \text{ in } \Omega, \\
 u(t=0) &= u_0 + \frac{z}{L_z} (u_1 - u_0) \\
 u|_{\Gamma_4} &= u_0, \quad u|_{\Gamma_2} = u_1, \quad \alpha(u - u_e) + \frac{\partial u}{\partial n}|_{\Gamma_1 \cup \Gamma_3} = 0.
 \end{aligned} \tag{3.3}$$

Note that the PDE has been multiplied by r .

After discretization in time with an implicit scheme, with time steps dt , in the FreeFem++ syntax r becomes x and z becomes y and the problem is:

```
problem thermaxi(u,v)=int2d(Th)((u*v/dt + dx(u)*dx(v) + dy(u)*dy(v))*x)
+ int1d(Th,3)(alpha*x*u*v) - int1d(Th,3)(alpha*x*ue*v)
- int2d(Th)(uold*v*x/dt) + on(2,4,u=u0);
```

Notice that the bilinear form degenerates at $x = 0$. Still one can prove existence and uniqueness for u and because of this degeneracy no boundary conditions need to be imposed on Γ_1 .

3.4.2 A Nonlinear Problem : Radiation

Heat loss through radiation is a loss proportional to the absolute temperature to the fourth power (Stefan's Law). This adds to the loss by convection and gives the following boundary condition:

$$\kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) + c[(u + 273)^4 - (u_e + 273)^4] = 0$$

The problem is nonlinear, and must be solved iteratively. If m denotes the iteration index, a semi-linearization of the radiation condition gives

$$\frac{\partial u^{m+1}}{\partial n} + \alpha(u^{m+1} - u_e) + c(u^{m+1} - u_e)(u^m + u_e + 546)((u^m + 273)^2 + (u_e + 273)^2) = 0,$$

because we have the identity $a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$. The iterative process will work with $v = u - u_e$.

```
...
fespace Vh(Th,P1);
real rad=1e-8, uek=ue+273;
Vh vold,w,v=u0-ue,b;
problem thermradia(v,w)
= int2d(Th)(v*w/dt + k*(dx(v) * dx(w) + dy(v) * dy(w)))
+ int1d(Th,1,3)(b*v*w)
- int2d(Th)(vold*w/dt) + on(2,4,v=u0-ue);

for(real t=0;t<T;t+=dt){
  vold=v;
  for(int m=0;m<5;m++){
    b= alpha + rad * (v + 2*uek) * ((v+uek)^2 + uek^2);
    thermradia;
  }
}
vold=v+ue; plot(vold);
```

3.5 Irrotational Fan Blade Flow and Thermal effects

Summary *Here we will learn how to deal with a multi-physics system of PDEs on a Complex geometry, with multiple meshes within one problem. We also learn how to manipulate the region indicator and see how smooth is the projection operator from one mesh to another.*

Incompressible flow Without viscosity and vorticity incompressible flows have a velocity given by:

$$u = \begin{pmatrix} \frac{\partial \psi}{\partial x_2} \\ -\frac{\partial \psi}{\partial x_1} \end{pmatrix}, \quad \text{where } \psi \text{ is solution of } \Delta \psi = 0$$

This equation expresses both incompressibility ($\nabla \cdot u = 0$) and absence of vortex ($\nabla \times u = 0$). As the fluid slips along the walls, normal velocity is zero, which means that ψ satisfies:

$$\psi \text{ constant on the walls.}$$

One can also prescribe the normal velocity at an artificial boundary, and this translates into non constant Dirichlet data for ψ .

Airfoil Let us consider a wing profile S in a uniform flow. Infinity will be represented by a large circle C where the flow is assumed to be of uniform velocity; one way to model this problem is to write

$$\Delta \psi = 0 \text{ in } \Omega, \quad \psi|_S = 0, \quad \psi|_C = u_\infty y, \quad (3.4)$$

where $\partial\Omega = C \cup S$

The NACA0012 Airfoil An equation for the upper surface of a NACA0012 (this is a classical wing profile in aerodynamics) is:

$$y = 0.17735\sqrt{x} - 0.075597x - 0.212836x^2 + 0.17363x^3 - 0.06254x^4.$$

Example 3.5 (potential.edp)

// file potential.edp

```
real S=99;
border C(t=0,2*pi) { x=5*cos(t); y=5*sin(t); }
border Splus(t=0,1){ x = t; y = 0.17735*sqrt(t)-0.075597*t
- 0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4); label=S; }
border Sminus(t=1,0){ x =t; y= -(0.17735*sqrt(t)-0.075597*t
-0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4)); label=S; }
mesh Th= buildmesh(C(50)+Splus(70)+Sminus(70));
fespace Vh(Th,P2); Vh psi,w;

solve potential(psi,w)=int2d(Th) (dx(psi)*dx(w)+dy(psi)*dy(w)) +
on(C,psi = y) + on(S,psi=0);

plot(psi,wait=1);
```

A zoom of the streamlines are shown on Figure 3.5.

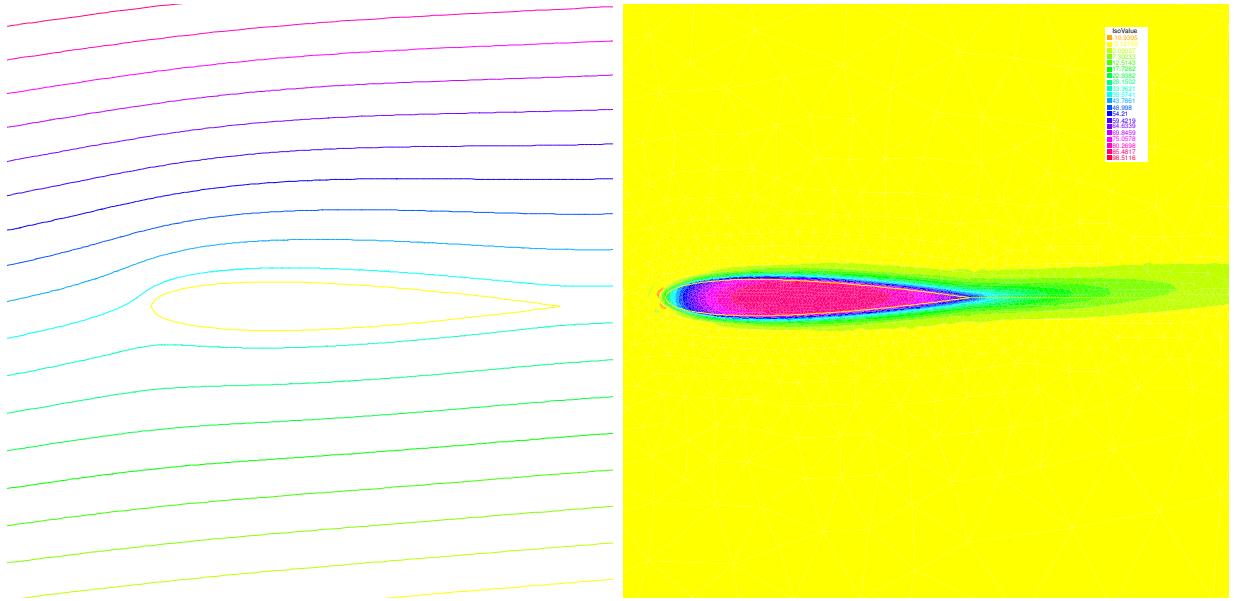


Figure 3.5: Zoom around the NACA0012 airfoil showing the streamlines (curve $\psi = \text{constant}$). To obtain such a plot use the interactive graphic command: “+” and p. Right: temperature distribution at time $T=25$ (now the maximum is at 90 instead of 120). Note that an incidence angle has been added here (see Chapter 9).

3.5.1 Heat Convection around the airfoil

Now let us assume that the airfoil is hot and that air is there to cool it. Much like in the previous section the heat equation for the temperature v is

$$\partial_t v - \nabla \cdot (\kappa \nabla v) + u \cdot \nabla v = 0, \quad v(t=0) = v_0, \quad \frac{\partial v}{\partial n}|_C = 0$$

But now the domain is outside AND inside S and κ takes a different value in air and in steel. Furthermore there is convection of heat by the flow, hence the term $u \cdot \nabla v$ above. Consider the following, to be plugged at the end of the previous program:

```
...
border D(t=0,2){x=1+t;y=0;} // added to have a fine mesh at trail
mesh Sh = buildmesh(C(25)+Splus(-90)+Sminus(-90)+D(200));
fespace Wh(Sh,P1); Wh v,vv;
int steel=Sh(0.5,0).region, air=Sh(-1,0).region;
fespace W0(Sh,P0);
W0 k=0.01*(region==air)+0.1*(region==steel);
W0 u1=dy(psi)*(region==air), u2=-dx(psi)*(region==air);
Wh vold = 120*(region==steel);
real dt=0.05, nbT=50;
int i;
problem thermic(v,vv,init=i,solver=LU)= int2d(Sh)(v*vv/dt
+ k*(dx(v)*dx(vv)+dy(v)*dy(vv))
+ 10*(u1*dx(v)+u2*dy(v))*vv)- int2d(Sh)(vold*vv/dt);
for(i=0;i<nbT;i++){
    v=vold; thermic;
```



```
plot(v);
}
```

Notice here

- how steel and air are identified by the mesh parameter region which is defined when buildmesh is called and takes an integer value corresponding to each connected component of Ω ;
- how the convection terms are added without upwinding. Upwinding is necessary when the Pecley number $|u|L/\kappa$ is large (here is a typical length scale), The factor 10 in front of the convection terms is a quick way of multiplying the velocity by 10 (else it is too slow to see something).
- The solver is Gauss' LU factorization and when `init` $\neq 0$ the LU decomposition is reused so it is much faster after the first iteration.

3.6 Pure Convection : The Rotating Hill

Summary *Here we will present two methods for upwinding for the simplest convection problem. We will learn about Characteristics-Galerkin and Discontinuous-Galerkin Finite Element Methods.*

Let Ω be the unit disk centered at 0; consider the rotation vector field

$$\mathbf{u} = [u_1, u_2], \quad u_1 = y, \quad u_2 = -x.$$

Pure convection by \mathbf{u} is

$$\partial_t c + \mathbf{u} \cdot \nabla c = 0 \quad \text{in } \Omega \times (0, T) \quad c(t=0) = c^0 \quad \text{in } \Omega.$$

The exact solution $c(x_t, t)$ at time t en point x_t is given by

$$c(x_t, t) = c^0(x, 0)$$

where x_t is the particle path in the flow starting at point x at time 0. So x_t are solutions of

$$\dot{x}_t = u(x_t), \quad , \quad x_{t=0} = x, \quad \text{where} \quad \dot{x}_t = \frac{d(t \mapsto x_t)}{dt}$$

The ODE are reversible and we want the solution at point x at time t (not at point x_t) the initial point is x_{-t} , and we have

$$c(x, t) = c^0(x_{-t}, 0)$$

The game consists in solving the equation until $T = 2\pi$, that is for a full revolution and to compare the final solution with the initial one; they should be equal.

Solution by a Characteristics-Galerkin Method In FreeFem++ there is an operator called `convect([u1,u2],dt,c)` which compute $c \circ X$ with X is the convect field defined by $X(x) = x_{dt}$ and where x_τ is particule path in the steady state velocity field $\mathbf{u} = [u1, u2]$ starting at point x at time $\tau = 0$, so x_τ is solution of the following ODE:

$$\dot{x}_\tau = u(x_\tau), \quad \mathbf{x}_{\tau=0} = x.$$

When \mathbf{u} is piecewise constant; this is possible because x_τ is then a polygonal curve which can be computed exactly and the solution exists always when u is divergence free; `convect` returns $c(x_{df}) = C \circ X$.

Example 3.6 (convects.edp)

// file convects.edp

```
border C(t=0, 2*pi) { x=cos(t); y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Uh(Th,P1);
Uh cold, c = exp(-10*((x-0.3)^2 + (y-0.3)^2));

real dt = 0.17, t=0;
Uh u1 = y, u2 = -x;
for (int m=0; m<2*pi/dt ; m++) {
  t += dt;      cold=c;
  c=convect([u1,u2],-dt,cold);
  plot(c, cmm=" t="+t + ", min=" + c[].min + ", max=" + c[].max);
}
```

Remark 4 3D plots can be done by adding the qualifier "dim=3" to the plot instruction.

The method is very powerful but has two limitations: a/ it is not conservative, b/ it may diverge in rare cases when $|u|$ is too small due to quadrature error.

Solution by Discontinuous-Galerkin FEM Discontinuous Galerkin methods take advantage of the discontinuities of c at the edges to build upwinding. There are may formulations possible. We shall implement here the so-called dual- P_1^{DC} formulation (see Ern[11]):

$$\int_{\Omega} \left(\frac{c^{n+1} - c^n}{\delta t} + u \cdot \nabla c \right) w + \int_E (\alpha |n \cdot u| - \frac{1}{2} n \cdot u) [c] w = \int_{E_T^-} |n \cdot u| c w \quad \forall w$$

where E is the set of inner edges and E_T^- is the set of boundary edges where $u \cdot n < 0$ (in our case there is no such edges). Finally $[c]$ is the jump of c across an edge with the convention that c^+ refers to the value on the right of the oriented edge.

Example 3.7 (convects_end.edp)

// file convects.edp

```
...
fespace Vh(Th,P1dc);

Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*((x-0.3)^2 + (y-0.3)^2));
real u, al=0.5; dt = 0.05;
```

```

macro n() (N.x*v1+N.y*v2) // Macro without parameter
problem Adu1(cc,w) =
int2d(Th) ((cc/dt+(v1*dx(cc)+v2*dy(cc)))*w)
+ intalledges(Th) ((1-nTonEdge)*w*(al*abs(n)-n/2)*jump(cc))
// - int1d(Th,C) ((n<0)*abs(n)*cc*w) // unused because cc=0 on ∂Ω⁻
- int2d(Th) (ccold*w/dt);

for ( t=0; t< 2*pi ; t+=dt)
{
  ccold=cc; Adu1;
  plot(cc,fill=1,cmm="t="+t + " , min=" + cc[].min + " , max=" + cc[].max);
};
real [int] viso=[-0.2,-0.1,0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1,1.1];
plot(c,wait=1,fill=1,ps="convectCG.eps",viso=viso);
plot(c,wait=1,fill=1,ps="convectDG.eps",viso=viso);

```

Notice the new keywords, `intalledges` to integrate on all edges of all triangles

$$\text{intalledges}(\text{Th}) \equiv \sum_{T \in \text{Th}} \int_{\partial T} \quad (3.5)$$

(so all internal edges are seen two times), `nTonEdge` which is one if the triangle has a boundary edge and two otherwise, `jump` to implement [c]. Results of both methods are shown on Figure 3.6 with identical levels for the level line; this is done with the plot-modifier `viso`.

Notice also the macro where the parameter u is not used (but the syntax needs one) and which ends with a `//`; it simply replaces the name `n` by `(N.x*v1+N.y*v2)`. As easily guessed `N.x`, `N.y` is the normal to the edge.

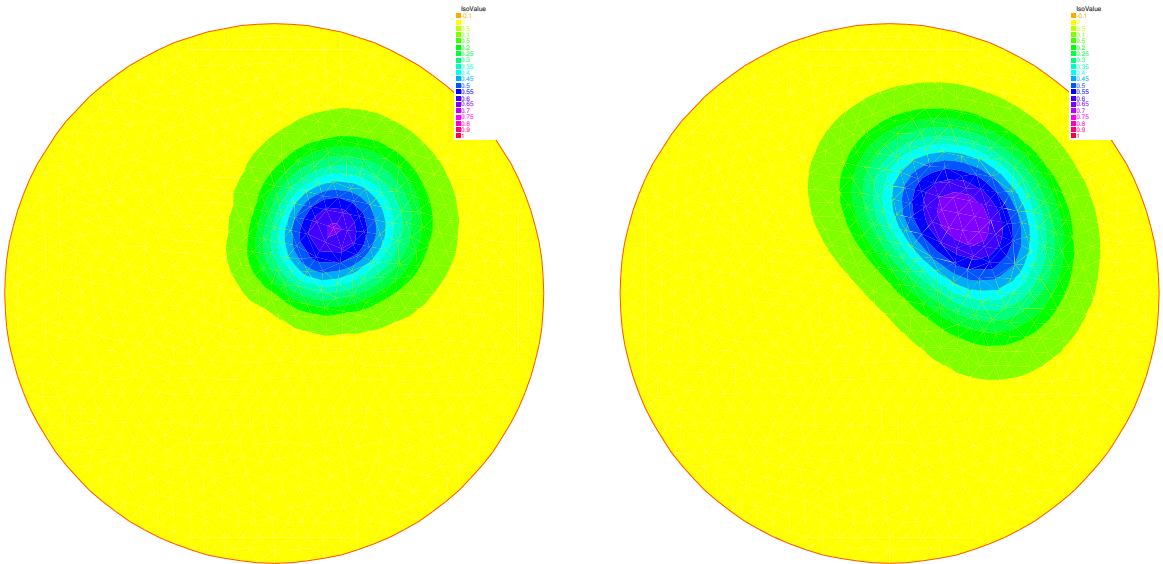


Figure 3.6: The rotated hill after one revolution, left with Characteristics-Galerkin, on the right with Discontinuous P_1 Galerkin FEM.

Now if you think that DG is too slow try this

```

//      the same DG very much faster
varf aadual(cc,w) = int2d(Th) ((cc/dt+(v1*dx(cc)+v2*dy(cc)))*w)
    + intalldedges(Th) ((1-nTonEdge)*w*(al*abs(n)-n/2)*jump(cc));
varf bbdual(ccold,w) = - int2d(Th) (ccold*w/dt);
matrix AA= aadual(Vh,Vh);
matrix BB = bbdual(Vh,Vh);
set (AA,init=t,solver=sparsesolver);
Vh rhs=0;
for ( t=0; t< 2*pi ; t+=dt)
{
    ccold=cc;
    rhs[] = BB* ccold[];
    cc[] = AA^-1*rhs[];
    plot(cc,fill=0,cmm="t="+t + ", min=" + cc[].min + ", max=" + cc[].max);
};

```

Notice the new keyword `set` to specify a solver in this framework; the modifier `init` is used to tell the solver that the matrix has not changed (`init=true`), and the name parameter are the same that in problem definition (see. 6.9) .

Finite Volume Methods can also be handled with `FreeFem++` but it requires programming. For instance the $P_0 - P_1$ Finite Volume Method of Dervieux et al associates to each P_0 function c^1 a P_0 function c^0 with constant value around each vertex q^i equal to $c^1(q^i)$ on the cell σ_i made by all the medians of all triangles having q^i as vertex. Then upwinding is done by taking left or right values at the median:

$$\int_{\sigma_i} \frac{1}{\delta t} (c^{1n+1} - c^{1n}) + \int_{\partial\sigma_i} u \cdot nc^- = 0 \quad \forall i$$

It can be programmed as

```

load "mat_dervieux"; //      external module in C++ must be loaded
border a(t=0, 2*pi){ x = cos(t); y = sin(t); }
mesh th = buildmesh(a(100));
fespace Vh(th,P1);

Vh vh,vold,u1 = y, u2 = -x;
Vh v = exp(-10*((x-0.3)^2 +(y-0.3)^2)), vWall=0, rhs =0;

real dt = 0.025;
//      qflpTlump means mass lumping is used
problem FVM(v,vh) = int2d(th,qft=qflpTlump)(v*vh/dt)
    - int2d(th,qft=qflpTlump)(vold*vh/dt)
    + int1d(th,a) (((u1*N.x+u2*N.y)<0)*(u1*N.x+u2*N.y)*vWall*vh)
+ rhs[] ;

matrix A;
MatUpWind0(A,th,vold,[u1,u2]);

for ( int t=0; t< 2*pi ; t+=dt){
    vold=v;
    rhs[] = A * vold[] ; FVM;
}

```

```
plot(v,wait=0);
};
```

the mass lumping parameter forces a quadrature formula with Gauss points at the vertices so as to make the mass matrix diagonal; the linear system solved by a conjugate gradient method for instance will then converge in one or two iterations.

The right hand side rhs is computed by an external C++ function `MatUpWind0(...)` which is programmed as

```

//      computes matrix a on a triangle for the Dervieux FVM
int  fvmP1P0(double q[3][2],           //      the 3 vertices of a triangle T
            double u[2],               //      convection velocity on T
            double c[3],               //      the P1 function on T
            double a[3][3],           //      output matrix
            double where[3] )         //      where>0 means we're on the boundary
{
    for(int i=0;i<3;i++) for(int j=0;j<3;j++) a[i][j]=0;

    for(int i=0;i<3;i++){
        int ip = (i+1)%3, ipp = (ip+1)%3;
        double unL = -((q[ip][1]+q[i][1]-2*q[ipp][1])*u[0]
                        -(q[ip][0]+q[i][0]-2*q[ipp][0])*u[1])/6;
        if(unL>0) { a[i][i] += unL; a[ip][i]-=unL; }
        else{ a[i][ip] += unL; a[ip][ip]-=unL; }
        if(where[i]&&where[ip]){ //      this is a boundary edge
            unL=((q[ip][1]-q[i][1])*u[0] - (q[ip][0]-q[i][0])*u[1])/2;
            if(unL>0) { a[i][i]+=unL; a[ip][ip]+=unL; }
        }
    }
    return 1;
}
```

It must be inserted into a larger .cpp file, shown in Appendix A, which is the load module linked to FreeFem++ .

3.7 The System of elasticity

Elasticity Solid objects deform under the action of applied forces: a point in the solid, originally at (x, y, z) will come to (X, Y, Z) after some time; the vector $\mathbf{u} = (u_1, u_2, u_3) = (X - x, Y - y, Z - z)$ is called the displacement. When the displacement is small and the solid is elastic, Hooke's law gives a relationship between the stress tensor $\sigma(u) = (\sigma_{ij}(u))$ and the strain tensor $\epsilon(u) = \epsilon_{ij}(u)$

$$\sigma_{ij}(u) = \lambda \delta_{ij} \nabla \cdot \mathbf{u} + 2\mu \epsilon_{ij}(u),$$

where the Kronecker symbol $\delta_{ij} = 1$ if $i = j$, 0 otherwise, with

$$\epsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

and where λ, μ are two constants that describe the mechanical properties of the solid, and are themselves related to the better known constants E , Young's modulus, and ν , Poisson's ratio:

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

Lamé's system Let us consider a beam with axis Oz and with perpendicular section Ω . The components along x and y of the strain $\mathbf{u}(x)$ in a section Ω subject to forces \mathbf{f} perpendicular to the axis are governed by

$$-\mu\Delta\mathbf{u} - (\mu + \lambda)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega,$$

where λ, μ are the Lamé coefficients introduced above.

Remark, we do not use this equation because the associated variational form does not give the right boundary condition, we simply use

$$-div(\sigma) = \mathbf{f} \quad \text{in } \Omega$$

where the corresponding variational form is:

$$\int_{\Omega} \sigma(u) : \epsilon(\mathbf{v}) \, dx - \int_{\Omega} \mathbf{v} f \, dx = 0;$$

where $:$ denote the tensor scalar product, i.e. $a : b = \sum_{i,j} a_{ij}b_{ij}$.
So the variational form can be written as :

$$\int_{\Omega} \lambda \nabla \cdot u \nabla \cdot v + 2\mu \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \, dx - \int_{\Omega} \mathbf{v} f \, dx = 0;$$

Example Consider elastic plate with the undeformed rectangle shape $[0, 20] \times [-1, 1]$. The body force is the gravity force \mathbf{f} and the boundary force \mathbf{g} is zero on lower, upper and right sides. The left vertical sides of the beam is fixed. The boundary conditions are

$$\begin{aligned} \sigma \cdot \mathbf{n} &= \mathbf{g} = 0 \quad \text{on } \Gamma_1, \Gamma_4, \Gamma_3, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_2 \end{aligned}$$

Here $\mathbf{u} = (u, v)$ has two components.

The above two equations are strongly coupled by their mixed derivatives, and thus any iterative solution on each of the components is risky. One should rather use FreeFem++'s system approach and write:

Example 3.8 (lame.edp)

```

mesh Th=square(10,10,[20*x,2*y-1]);
espace Vh(Th,P2);
Vh u,v,uu,vv;
real sqrt2=sqrt(2.);
macro epsilon(u1,u2) [dx(u1),dy(u2),(dy(u1)+dx(u2))/sqrt2]

```

// file lame.edp
// EOM

```

// the sqrt2 is because we want: epsilon(u1,u2)'* epsilon(v1,v2)
== epsilon(u):epsilon(v)
macro div(u,v) ( dx(u)+dy(v) ) // EOM

real E = 21e5, nu = 0.28, mu= E/(2*(1+nu));
real lambda = E*nu/((1+nu)*(1-2*nu)), f = -1; //

solve lame([u,v],[uu,vv])= int2d(Th) (
    lambda*div(u,v)*div(uu,vv)
    +2.*mu*( epsilon(u,v)'*epsilon(uu,vv) ) )
    - int2d(Th) (f*v)
    + on(4,u=0,v=0);
real coef=100;
plot ([u,v],wait=1,ps="lamevect.eps",coef=coef);

mesh th1 = movemesh(Th, [x+u*coef, y+v*coef]);
plot (th1,wait=1,ps="lamedeform.eps");
real dxmin = u[].min;
real dymin = v[].min;

cout << " - dep. max x = "<< dxmin<< " y=" << dymin << endl;
cout << " dep. (20,0) = " << u(20,0) << " " << v(20,0) << endl;

```

The numerical results are shown on figure 3.7 and the output is:

```

-- square mesh : nb vertices =121 , nb triangles = 200 , nb boundary edges 40
-- Solve : min -0.00174137 max 0.00174105
            min -0.0263154 max 1.47016e-29
- dep. max x = -0.00174137 y=-0.0263154
  dep. (20,0) = -1.8096e-07 -0.0263154
times: compile 0.010219s, execution 1.5827s

```

3.8 The System of Stokes for Fluids

In the case of a flow invariant with respect to the third coordinate (two-dimensional flow), flows at low Reynolds number (for instance micro-organisms) satisfy,

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

where $\mathbf{u} = (u_1, u_2)$ is the fluid velocity and p its pressure.

The driven cavity is a standard test. It is a box full of liquid with its lid moving horizontally at speed one. The pressure and the velocity must be discretized in compatible finite element spaces for the LBB conditions to be satisfied:

$$\sup_{p \in P_h} \frac{(\mathbf{u}, \nabla p)}{|p|} \geq \beta |\mathbf{u}| \quad \forall \mathbf{u} \in U_h$$

// file stokes.edp

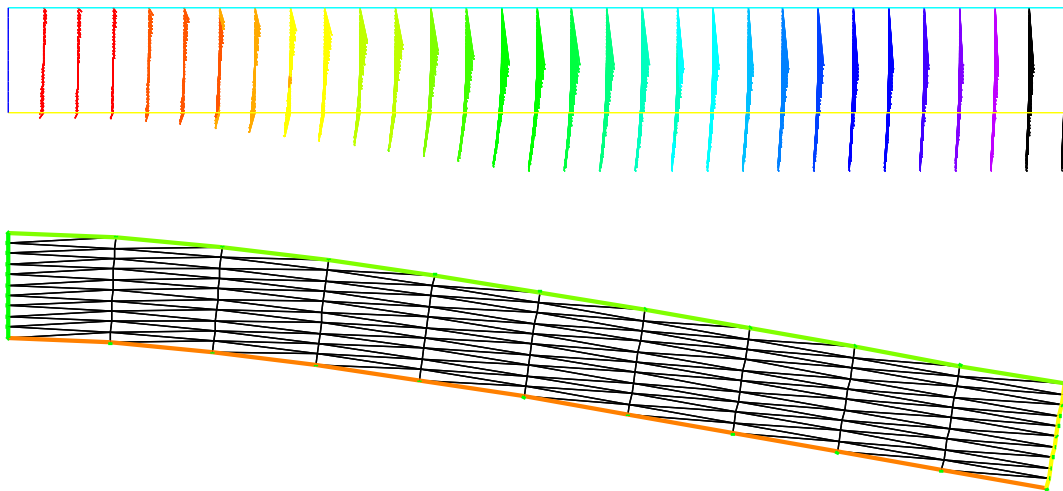


Figure 3.7: Solution of Lamé's equations for elasticity for a 2D beam deflected by its own weight and clamped by its left vertical side; result are shown with a amplification factor equal to 100. *Remark: the size of the arrow is automatically bound, but the color gives the real length*

```

int n=3;
mesh Th=square(10*n,10*n);
fespace Uh(Th,P1b); Uh u,v,uu,vv;
fespace Ph(Th,P1); Ph p,pp;

solve stokes([u,v,p],[uu,vv,pp]) =
  int2d(Th) (dx(u)*dx(uu)+dy(u)*dy(uu) + dx(v)*dx(vv)+ dy(v)*dy(vv)
    + dx(p)*uu + dy(p)*vv + pp*(dx(u)+dy(v))
    - 1e-10*p*pp)
  + on(1,2,4,u=0,v=0) + on(3,u=1,v=0);
plot([u,v],p,wait=1);

```

Remark, we add a stabilization term $-10e-10*p*pp$ to fixe the constant part of the pressure.

Results are shown on figure 3.8

3.9 A Projection Algorithm for the Navier-Stokes equations

Summary *Fluid flows require good algorithms and good triangultions. We show here an example of a complex algorithm and or first example of mesh adaptation.*

An incompressible viscous fluid satisfies:

$$\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = 0, \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times]0, T[,$$

$$u|_{t=0} = u^0, \quad u|_{\Gamma} = u_{\Gamma}.$$

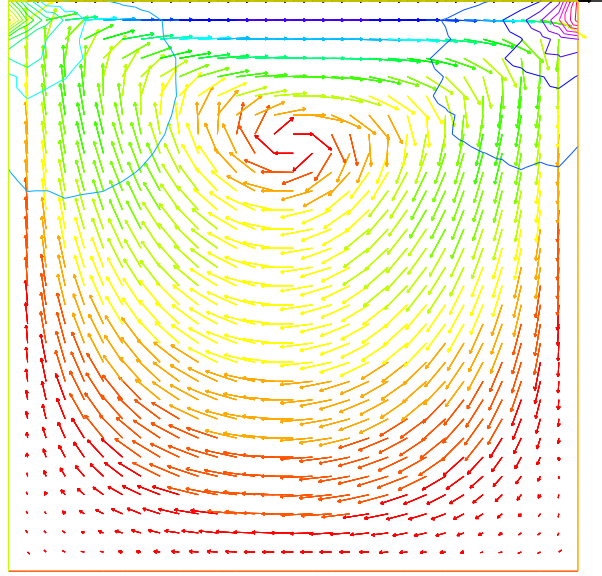


Figure 3.8: Solution of Stokes' equations for the driven cavity problem, showing the velocity field and the pressure level lines.

A possible algorithm, proposed by Chorin, is

$$\frac{1}{\delta t}[u^{m+1} - u^m \circ X^m] + \nabla p^m - \nu \Delta u^m = 0, \quad u|_{\Gamma} = u_{\Gamma}, \nu \partial_n u|_{\Gamma_{out}} = 0$$

$$-\Delta p^{m+1} = -\nabla \cdot u^m \circ X^m, \quad \partial_n p^{m+1} = 0 \text{ on } \Gamma, p^{m+1} = 0 \text{ on } \Gamma_{out}$$

where $u \circ X(x) = u(x - u(x)\delta t)$ since $\partial_t u + u \cdot \nabla u$ is approximated by the method of characteristics, as in the previous section.

We use the s Chorin's algorithm with free boundary condition at outlet (i.e. $p = 0, \nu \partial_n u = 0$), to compute a correction, q , to the pressure.

$$-\Delta q = \nabla \cdot \mathbf{u}, \quad q = 0 \text{ on } \Gamma_{out}$$

and define

$$u^{m+1} = \tilde{u} + P \nabla q \delta t, \quad p^{m+1} = p^m - q$$

where \tilde{u} is the (u^{m+1}, v^{m+1}) of Chorin's algorithm, and where P is the L^2 projection with mass lumping (a sparse matrix).

The backward facing step The geometry is that of a channel with a backward facing step so that the inflow section is smaller than the outflow section. This geometry produces a fluid recirculation zone that must be captured correctly.

This can only be done if the triangulation is sufficiently fine, or well adapted to the flow.

Remark (FH), The are a technical difficulty is the example, is the output B.C., here we put $p = 0$ and $\nu \partial_n u = 0$.

Example 3.9 (NSprojection.edp)

```

//      file NSprojection.edp
//      Version july 2014,
//      FH. Change B.C on u on output , more simple ..
//      .....

verbosity=0;
border a0(t=1,0){ x=-2;      y=t;      label=1;}
border a1(t=-2,0){ x=t;      y=0;      label=2;}
border a2(t=0,-0.5){ x=0;      y=t;      label=2;}
border a3(t=0,1){ x=18*t^1.2; y=-0.5;    label=2;}
border a4(t=-0.5,1){ x=18;      y=t;      label=3;}
border a5(t=1,0){ x=-2+20*t; y=1;      label=4;}
int n=1;
mesh Th= buildmesh(a0(3*n)+a1(20*n)+a2(10*n)+a3(150*n)+a4(5*n)+a5(100*n));

plot (Th);
fespace Vh(Th,P1);
real nu = 0.0025, dt = 0.2;

Vh w,u = 0, v =0, p = 0, q=0;

real epsv = 1e-6, epsu = 1e-6, epsp = 1e-6;

//      Reynolds=200

//      def of Matrix dtMx and dtMy

matrix dtM1x,dtM1y;

macro BuildMat()
{ /* for memory managenemt */
  varf vM(unused,v) = int2d(Th)(v) ;
  varf vdx(u,v) = int2d(Th)(v*dx(u)*dt) ;
  varf vdy(u,v) = int2d(Th)(v*dy(u)*dt) ;

  real[int] Mlump = vM(0,Vh);
  real[int] one(Vh.ndof); one = 1;
  real[int] M1 = one ./ Mlump;
  matrix dM1 = M1;
  matrix Mdx = vdx(Vh,Vh);
  matrix Mdy = vdy(Vh,Vh);
  dtM1x = dM1*Mdx;
  dtM1y = dM1*Mdy;
} //      EOF

BuildMat

real err=1, outflux=1;
for(int n=0;n<300;n++)
{
  Vh uold = u, vold = v, pold=p;

  solve pb4u(u,w,init=n,solver=CG,eps=epsu)
    =int2d(Th)(u*w/dt +nu*(dx(u)*dx(w)+dy(u)*dy(w)))
    -int2d(Th)((convect([uold,vold],-dt,uold)/dt-dx(p))*w)
    + on(1,u = 4*y*(1-y)) + on(2,4,u = 0) ;

  plot (u);

  solve pb4v(v,w,init=n,solver=CG,eps=epsv)

```

```

    = int2d(Th) (v*w/dt +nu*(dx(v)*dx(w)+dy(v)*dy(w)))
    -int2d(Th) ((convect([uold,vold],-dt,vold)/dt-dy(p))*w)
    +on(1,2,3,4,v = 0);

solve pb4p(q,w,solver=CG,init=n,eps=eps) = int2d(Th) (dx(q)*dx(w)+dy(q)*dy(w))
    - int2d(Th) ((dx(u)+ dy(v))*w/dt)+ on(3,q=0);

                                     //    to have absolute epsilon in CG algorithm.

    epsv = -abs(epsv);
    epsu = -abs(epsu);
    epsp = -abs(epsp);

    p = pold-q;
    u[] += dtMlx*q[];
    v[] += dtMly*q[];

    if (n%50==49) {
        Th = adaptmesh(Th, [u,v],q,err=0.04,nbvx=100000);
        plot(Th, wait=true);
        BuildMat                                     //    rebuild mat.
    }

    err = sqrt(int2d(Th) (square(u-uold)+square(v-vold))/Th.area) ;
    outflux = int1d(Th) ( [u,v]'*[N.x,N.y]) ;
    cout << " iter " << n << " Err L2 = " << err
        << " flux sortant = "<< outflux << endl;
    if (err < 1e-3) break;
}

assert (abs(outflux)< 2e-3);                                     //    verifaction ...
plot (p,wait=1,ps="NSprojP.eps");
plot (u,wait=1,ps="NSprojU.eps");

```

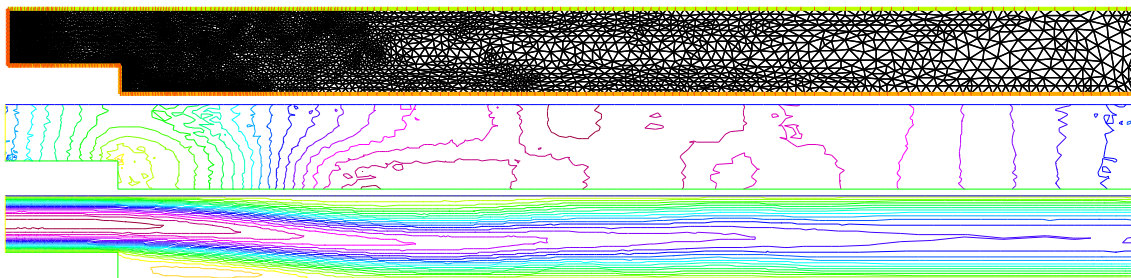


Figure 3.9: Rannacher's projection algorithm: result on an adapted mesh (top) showing the pressure (middle) and the horizontal velocity u at Reynolds 400.

We show in figure 3.9 the numerical results obtained for a Reynolds number of 400 where mesh adaptation is done after 50 iterations on the first mesh.

3.10 Newton Method for the Steady Navier-Stokes equations

The problem is find the velocity field $\mathbf{u} = (u_i)_{i=1}^d$ and the pressure p of a Flow satisfying in the domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$:

$$\begin{aligned} (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

where ν is the viscosity of the fluid, $\nabla = (\partial_i)_{i=1}^d$, the dot product is \cdot , and $\Delta = \nabla \cdot \nabla$ with the some boundary conditions (\mathbf{u} is given on Γ)

The weak form is find \mathbf{u}, p such than for $\forall \mathbf{v}$ (zero on Γ), and $\forall q$

$$\int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{v} + \nu \nabla \mathbf{u} : \nabla \mathbf{v} - p \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u} = 0 \quad (3.6)$$

The Newton Algorithm to solve nonlinear Problem is
Find $u \in V$ such that $F(u) = 0$ where $F : V \mapsto V$.

1. choose $u_0 \in \mathbb{R}^n$;
2. for ($i = 0$; $i \leq \text{niter}$; $i = i + 1$)

(a) solve $DF(u_i)w_i = F(u_i)$;

(b) $u_{i+1} = u_i - w_i$;

break $\|w_i\| < \varepsilon$.

Where $DF(u)$ is the differential of F at point u , this is a linear application such that:
 $F(u + \delta) = F(u) + DF(u)\delta + o(\delta)$
For Navier Stokes, F and DF are :

$$\begin{aligned} F(\mathbf{u}, p) &= \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{v} + \nu \nabla \mathbf{u} : \nabla \mathbf{v} - p \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u} \\ DF(\mathbf{u}, p)(\delta \mathbf{u}, \delta p) &= \int_{\Omega} ((\delta \mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{v} + ((\mathbf{u} \cdot \nabla) \delta \mathbf{u}) \cdot \mathbf{v} \\ &\quad + \nu \nabla \delta \mathbf{u} : \nabla \mathbf{v} - \delta p \nabla \cdot \mathbf{v} - q \nabla \cdot \delta \mathbf{u} \end{aligned}$$

So the Newton algorithm become

Example 3.10 (NSNewton.edp) ...

```
for( n=0; n< 15; n++)
{ solve Oseen([du1, du2, dp], [v1, v2, q]) =
    int2d(Th) ( nu*(Grad(du1, du2)'*Grad(v1, v2) )
                + UgradV(du1, du2, u1, u2)'*[v1, v2]
                + UgradV(u1, u2, du1, du2)'*[v1, v2]
                - div(du1, du2)*q - div(v1, v2)*dp
                - 1e-8*dp*q // stabilization term
    )
```

```

- int2d(Th) ( nu*(Grad(u1,u2)'*Grad(v1,v2) )
              + UgradV(u1,u2, u1, u2)'*[v1,v2]
              - div(u1,u2)*q - div(v1,v2)*p
            )
+ on(1,du1=0,du2=0) ;
u1[] -= du1[]; u2[] -= du2[]; p[] -= dp[];
err= du1[].linfty + du2[].linfty + dp[].linfty;
if(err < eps) break;
if( n>3 && err > 10.) break;                                     // blowup ???
}

```

With the operator:

```

macro Grad(u1,u2) [ dx(u1),dy(u1) , dx(u2),dy(u2) ] //
macro UgradV(u1,u2,v1,v2) [ [u1,u2]'*[dx(v1),dy(v1)] , //
                             [u1,u2]'*[dx(v2),dy(v2)] ] //
macro div(u1,u2) (dx(u1)+dy(u2)) //

```

We build a computation mesh the exterior of a 2d cylinder.

```

real R = 5,L=15;
border cc(t=0,2*pi){ x=cos(t)/2;y=sin(t)/2;label=1;}
border ce(t=pi/2,3*pi/2) { x=cos(t)*R;y=sin(t)*R;label=1;}
border beb(tt=0,1) { real t=tt^1.2; x= t*L; y= -R; label = 1;}
border beu(tt=1,0) { real t=tt^1.2; x= t*L; y= R; label = 1;}
border beo(t=-R,R) { x= L; y= t; label = 0;}
border bei(t=-R/4,R/4) { x= L/2; y= t; label = 0;}
mesh Th=buildmesh(cc(-50)+ce(30)+beb(20)+beu(20)+beo(10)+bei(10));
plot (Th);

// bounding box for the plot
func bb=[[-1,-2],[4,2]];

/ FE Space Taylor Hood
fespace Xh(Th,P2); // for volicity
fespace Mh(Th,P1); // for pressure
Xh u1,u2,v1,v2,du1,du2,u1p,u2p;
Mh p,q,dp,pp;

// intial guess with B.C.
u1 = ( x^2+y^2) > 2;
u2=0;

```

Finally we use trick to make continuation on the viscosity ν , because the Newton method blowup owe start with the final viscosity ν

```

// Physical parameter
real nu= 1./50, nufinal=1/200. ,cnu=0.5;

// stop test for Newton
real eps=1e-6;

verbosity=0;
while(1) // Loop on viscosity
{ int n;
  real err=0; // err on Newton algo ...

```

```

... put the new the Newton algo here

if(err < eps)
{
    // converge decrease  $\nu$  (more difficult)
    plot([u1,u2],p,wait=1,cmm=" rey = " + 1./nu , coef=0.3,bb=bb);
    if( nu == nufinal) break;
    if( n < 4) cnu=cnu^1.5; // fast converge => change faster
    nu = max(nufinal, nu* cnu); // new viscosity
    u1p=u1; u2p=u2; pp=p; // save correct solution ...
}
else
{
    // blowup increase  $\nu$  (more simple)
    // the method finally blowup
    // get previous value of viscosity
    // no conv. => change lower
    // new viscosity
    cout << " restart nu = " << nu << " Rey= "<< 1./nu << " (cnu = " << cnu << "
) \n";

    // restore a correct solution ..

    u1=u1p;
    u2=u2p;
    p=pp;
}
}

```

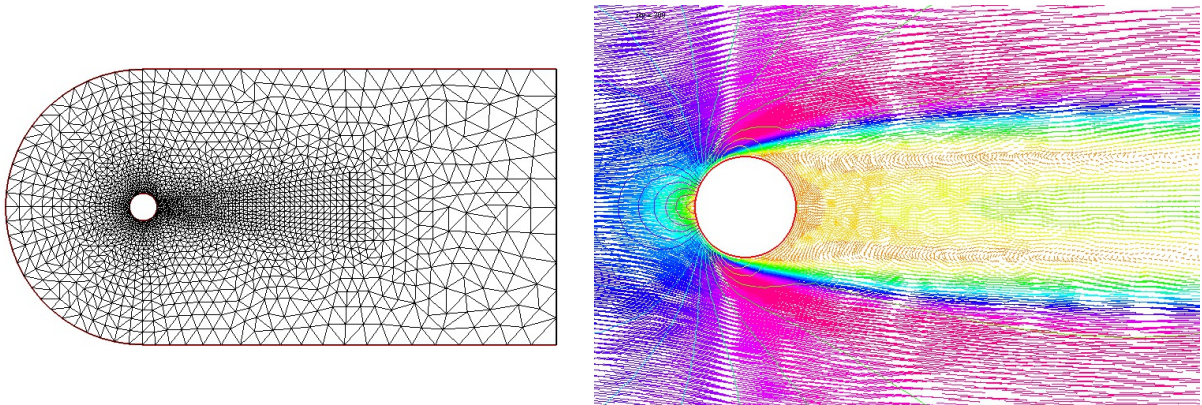


Figure 3.10: Mesh and the velocity and pressure at Reynolds 200

3.11 A Large Fluid Problem

A friend of one of us in Auroville-India was building a ramp to access an air conditioned room. As I was visiting the construction site he told me that he expected to cool air escaping by the door to the room to slide down the ramp and refrigerate the feet of the coming visitors. I told him "no way" and decided to check numerically. The results are on the front page of this book.

The fluid velocity and pressure are solution of the Navier-Stokes equations with varying

density function of the temperature.

The geometry is trapezoidal with prescribed inflow made of cool air at the bottom and warm air above and so are the initial conditions; there is free outflow, slip velocity at the top (artificial) boundary and no-slip at the bottom. However the Navier-Stokes cum temperature equations have a RANS $k - \epsilon$ model and a Boussinesq approximation for the buoyancy. This comes to

$$\begin{aligned}
\partial_t \theta + u \nabla \theta - \nabla \cdot (\kappa_T^m \nabla \theta) &= 0 \\
\partial_t u + u \nabla u - \nabla \cdot (\mu_T \nabla u) + \nabla p + e(\theta - \theta_0) \mathbf{e}_2, \quad \nabla \cdot u &= 0 \\
\mu_T &= c_\mu \frac{k^2}{\epsilon}, \quad \kappa_T = \kappa \mu_T \\
\partial_t k + u \nabla k + \epsilon - \nabla \cdot (\mu_T \nabla k) &= \frac{\mu_T}{2} |\nabla u + \nabla u^T|^2 \\
\partial_t \epsilon + u \nabla \epsilon + c_2 \frac{\epsilon^2}{k} - \frac{c_\epsilon}{c_\mu} \nabla \cdot (\mu_T \nabla \epsilon) &= \frac{c_1}{2} k |\nabla u + \nabla u^T|^2 = 0
\end{aligned} \tag{3.7}$$

We use a time discretization which preserves positivity and uses the method of characteristics ($X^m(x) \approx x - u^m(x) \delta t$)

$$\begin{aligned}
\frac{1}{\delta t} (\theta^{m+1} - \theta^m \circ X^m) - \nabla \cdot (\kappa_T^m \nabla \theta^{m+1}) &= 0 \\
\frac{1}{\delta t} (u^{m+1} - u^m \circ X^m) - \nabla \cdot (\mu_T^m \nabla u^{m+1}) + \nabla p^{m+1} + e(\theta^{m+1} - \theta_0) \mathbf{e}_2, \quad \nabla \cdot u^{m+1} &= 0 \\
\frac{1}{\delta t} (k^{m+1} - k^m \circ X^m) + k^{m+1} \frac{\epsilon^m}{k^m} - \nabla \cdot (\mu_T^m \nabla k^{m+1}) &= \frac{\mu_T^m}{2} |\nabla u^m + \nabla u^{mT}|^2 \\
\frac{1}{\delta t} (\epsilon^{m+1} - \epsilon^m \circ X^m) + c_2 \epsilon^{m+1} \frac{\epsilon^m}{k^m} - \frac{c_\epsilon}{c_\mu} \nabla \cdot (\mu_T^m \nabla \epsilon^{m+1}) &= \frac{c_1}{2} k^m |\nabla u^m + \nabla u^{mT}|^2 \\
\mu_T^{m+1} &= c_\mu \frac{k^{m+12}}{\epsilon^{m+1}}, \quad \kappa_T^{m+1} = \kappa \mu_T^{m+1}
\end{aligned} \tag{3.8}$$

In variational form and with appropriated boundary conditions the problem is:

```

real L=6;
border aa(t=0,1){x=t; y=0 ;}
border bb(t=0,14){x=1+t; y= - 0.1*t ;}
border cc(t=-1.4,L){x=15; y=t ;}
border dd(t=15,0){x= t ; y = L;}
border ee(t=L,0.5){ x=0; y=t ;}
border ff(t=0.5,0){ x=0; y=t ;}
int n=8;
mesh Th=buildmesh(aa(n)+bb(9*n) + cc(4*n) + dd(10*n)+ee(6*n) + ff(n));
real s0=clock();

fespace Vh2(Th,P1b); // velocity space
fespace Vh(Th,P1); // pressure space
fespace V0h(Th,P0); // for gradients
Vh2 u2,v2,up1=0,up2=0;
Vh2 u1,v1;
Vh u1x=0,u1y,u2x,u2y, vv;

real reynods=500;
// cout << " Enter the reynolds number :"; cin >> reynods;
assert(reynods>1 && reynods < 100000);

```

```

up1=0;
up2=0;
func g=(x)*(1-x)*4; // inflow
Vh p=0,q, temp1,temp=35, k=0.001,k1,ep=0.0001,ep1;
V0h muT=1,prodk,prode, kappa=0.25e-4, stress;
real alpha=0, eee=9.81/303, c1m = 1.3/0.09 ;
real nu=1, numu=nu/sqrt( 0.09), nuep=pow(nu,1.5)/4.1;
int i=0,iter=0;
real dt=0;
problem TEMPER(temp,q) = // temperature equation
    int2d(Th) (
        alpha*temp*q + kappa * ( dx(temp)*dx(q) + dy(temp)*dy(q) ))
        // + int1d(Th,aa,bb) (temp*q* 0.1)
    + int2d(Th) ( -alpha*convect([up1,up2],-dt,temp1)*q )
    + on(ff,temp=25)
    + on(aa,bb,temp=35) ;

problem kine(k,q)= // get the kinetic turbulent energy
    int2d(Th) (
        (ep1/k1+alpha)*k*q + muT * ( dx(k)*dx(q) + dy(k)*dy(q) ))
        // + int1d(Th,aa,bb) (temp*q*0.1)
    + int2d(Th) ( prodk*q-alpha*convect([up1,up2],-dt,k1)*q )
    + on(ff,k=0.0001) + on(aa,bb,k=numu*stress) ;

problem viscturb(ep,q)= // get the rate of turbulent viscous energy
    int2d(Th) (
        (1.92*ep1/k1+alpha)*ep*q + c1m*muT * ( dx(ep)*dx(q) + dy(ep)*dy(q)
    ))
        // + int1d(Th,aa,bb) (temp*q*0.1)
    + int2d(Th) ( prode*q-alpha*convect([up1,up2],-dt,ep1)*q )
    + on(ff,ep= 0.0001) + on(aa,bb,ep=nuep*pow(stress,1.5)) ;

solve NS ([u1,u2,p],[v1,v2,q]) = // Navier-Stokes k-epsilon and Boussinesq
    int2d(Th) (
        alpha*( u1*v1 + u2*v2)
        + muT * (dx(u1)*dx(v1)+dy(u1)*dy(v1)+dx(u2)*dx(v2)+dy(u2)*dy(v2))
        // ( 2*dx(u1)*dx(v1) + 2*dy(u2)*dy(v2)+(dy(u1)+dx(u2)) *(dy(v1)+dx(v2)) )
        + p*q*(0.000001)
        - p*dx(v1) - p*dy(v2)
        - dx(u1)*q - dy(u2)*q
    )
    + int1d(Th,aa,bb,dd) (u1*v1* 0.1)
    + int2d(Th) (eee*(temp-35)*v1 -alpha*convect([up1,up2],-dt,up1)*v1
        -alpha*convect([up1,up2],-dt,up2)*v2 )
    + on(ff,u1=3,u2=0)
    + on(ee,u1=0,u2=0)
    + on(aa,dd,u2=0)
    + on(bb,u2= -up1*N.x/N.y)
    + on(cc,u2=0) ;
plot(coef=0.2,cmm=" [u1,u2] et p ",p,[u1,u2],ps="StokesP2P1.eps",value=1,wait=1);
{
    real[int] xx(21),yy(21),pp(21);
    for (int i=0;i<21;i++)
    {
        yy[i]=i/20.;
    }
}

```



```

    xx[i]=u1(0.5,i/20.);
    pp[i]=p(i/20.,0.999);
}
    cout << " " << yy << endl;
                                //      plot([xx,yy],wait=1,cmm="u1 x=0.5 cup");
                                //      plot([yy,pp],wait=1,cmm="pressure y=0.999 cup");
}

dt = 0.05;
int nbiter = 3;
real coefdt = 0.25^(1./nbiter);
real coefcut = 0.25^(1./nbiter) , cut=0.01;
real tol=0.5,coefitol = 0.5^(1./nbiter);
nu=1./reynods;

for (iter=1;iter<=nbiter;iter++)
{
    cout << " dt = " << dt << " ----- " << endl;
    alpha=1/dt;
    for (i=0;i<=500;i++)
    {
        up1=u1;
        up2=u2;
        temp1=max(temp,25);
        temp1=min(temp1,35);
        k1=k; ep1=ep;
        muT=0.09*k*k/ep;
        NS; plot([u1,u2],wait=1); //      Solves Navier-Stokes
        prode =0.126*k*(pow(2*dx(u1),2)+pow(2*dy(u2),2)+2*pow(dx(u2)+dy(u1),2))/2;
        prodk= prode*k/ep*0.09/0.126;
        kappa=muT/0.41;
        stress=abs(dy(u1));
        kine; plot(k,wait=1);
        viscturb; plot(ep,wait=1);
        TEMPER; //      solves temperature equation
        if ( !(i % 5)){
            plot(temp,value=1,fill=true,ps="temp_"+iter+"_"+i+".ps");
            plot(coef=0.2,cmm=" [u1,u2] et p ",p,[u1,u2],ps="plotNS_"+iter+"_"+i+".ps");
        }
        cout << "CPU " << clock()-s0 << "s " << endl;
    }

    if (iter>= nbiter) break;
    Th=adaptmesh(Th,[dx(u1),dy(u1),dx(u1),dy(u2)],splitpbedge=1,
        abserror=0,cutoff=cut,err=tol, inquire=0,ratio=1.5,hmin=1./1000);
    plot(Th,ps="ThNS.eps");
    dt = dt*coefdt;
    tol = tol *coefitol;
    cut = cut *coefcut;
}
cout << "CPU " <<clock()-s0 << "s " << endl;

```

3.12 An Example with Complex Numbers

In a microwave oven heat comes from molecular excitation by an electromagnetic field. For a plane monochromatic wave, amplitude is given by Helmholtz's equation:

$$\beta v + \Delta v = 0.$$

We consider a rectangular oven where the wave is emitted by part of the upper wall. So the boundary of the domain is made up of a part Γ_1 where $v = 0$ and of another part $\Gamma_2 = [c, d]$ where for instance $v = \sin(\pi \frac{y-c}{c-d})$.

Within an object to be cooked, denoted by B , the heat source is proportional to v^2 . At equilibrium, one has

$$-\Delta\theta = v^2 I_B, \quad \theta_{\Gamma} = 0$$

where I_B is 1 in the object and 0 elsewhere.

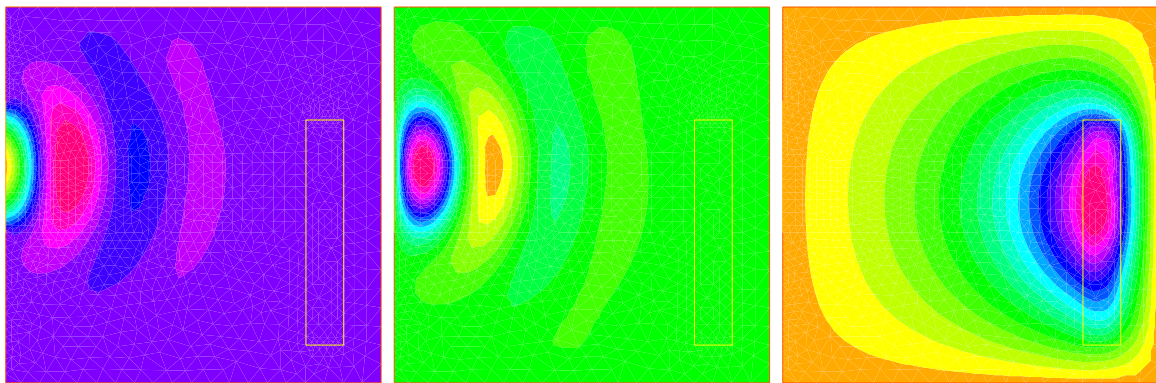


Figure 3.11: A microwave oven: real (left) and imaginary (middle) parts of wave and temperature (right).

Results are shown on figure 3.11

In the program below $\beta = 1/(1 - I/2)$ in the air and $2/(1 - I/2)$ in the object ($i = \sqrt{-1}$):

Example 3.11 (muwave.edp)

```

real a=20, b=20, c=15, d=8, e=2, l=12, f=2, g=2;
border a0(t=0,1) {x=a*t; y=0; label=1;}
border a1(t=1,2) {x=a; y= b*(t-1); label=1;}
border a2(t=2,3) { x=a*(3-t); y=b; label=1;}
border a3(t=3,4) {x=0; y=b-(b-c)*(t-3); label=1;}
border a4(t=4,5) {x=0; y=c-(c-d)*(t-4); label=2;}
border a5(t=5,6) { x=0; y= d*(6-t); label=1;}

border b0(t=0,1) {x=a-f+e*(t-1); y=g; label=3;}
border b1(t=1,4) {x=a-f; y=g+l*(t-1)/3; label=3;}
border b2(t=4,5) {x=a-f-e*(t-4); y=l+g; label=3;}
border b3(t=5,8) {x=a-e-f; y= l+g-l*(t-5)/3; label=3;}
int n=2;

```

// file muwave.edp

```

mesh Th = buildmesh(a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n)
                    +a4(10*n)+a5(10*n)+b0(5*n)+b1(10*n)+b2(5*n)+b3(10*n));
plot(Th,wait=1);
fespace Vh(Th,P1);
real meat = Th(a-f-e/2,g+l/2).region, air= Th(0.01,0.01).region;
Vh R=(region-air)/(meat-air);

Vh<complex> v,w;
solve muwave(v,w) = int2d(Th)(v*w*(1+R)
                        - (dx(v)*dx(w)+dy(v)*dy(w))*(1-0.5i))
  + on(1,v=0) + on(2, v=sin(pi*(y-c)/(c-d)));
Vh vr=real(v), vi=imag(v);
plot(vr,wait=1,ps="rmuonde.ps", fill=true);
plot(vi,wait=1,ps="imuonde.ps", fill=true);

fespace Uh(Th,P1); Uh u,uu, ff=1e5*(vr^2 + vi^2)*R;

solve temperature(u,uu)= int2d(Th)(dx(u)* dx(uu)+ dy(u)* dy(uu))
  - int2d(Th)(ff*uu) + on(1,2,u=0);
plot(u,wait=1,ps="tempmuonde.ps", fill=true);

```

3.13 Optimal Control

Thanks to the function BFGS it is possible to solve complex nonlinear optimization problem within FreeFem++ . For example consider the following inverse problem

$$\min_{b,c,d \in R} J = \int_E (u - u_d)^2 : -\nabla(\kappa(b, c, d) \cdot \nabla u) = 0, \quad u|_{\Gamma} = u_{\Gamma}$$

where the desired state u_d , the boundary data u_{Γ} and the observation set $E \subset \Omega$ are all given. Furthermore let us assume that

$$\kappa(x) = 1 + bI_B(x) + cI_C(x) + dI_D(x) \quad \forall x \in \Omega$$

where B, C, D are separated subsets of Ω .

To solve this problem by the quasi-Newton BFGS method we need the derivatives of J with respect to b, c, d . We self explanatory notations, if $\delta b, \delta c, \delta d$ are variations of b, c, d we have

$$\delta J \approx 2 \int_E (u - u_d) \delta u, \quad -\nabla(\kappa \cdot \nabla \delta u) \approx \nabla(\delta \kappa \cdot \nabla u) \quad \delta u|_{\Gamma} = 0$$

Obviously J'_b is equal to δJ when $\delta b = 1, \delta c = 0, \delta d = 0$, and so on for J'_c and J'_d .

All this is implemented in the following program

```

// file optimcontrol.edp
border aa(t=0, 2*pi) { x = 5*cos(t); y = 5*sin(t); };
border bb(t=0, 2*pi) { x = cos(t); y = sin(t); };
border cc(t=0, 2*pi) { x = -3+cos(t); y = sin(t); };
border dd(t=0, 2*pi) { x = cos(t); y = -3+sin(t); };

```

```

mesh th = buildmesh(aa(70)+bb(35)+cc(35)+dd(35));
fespace Vh(th,P1);
Vh Ib=((x^2+y^2)<1.0001),
    Ic=((x+3)^2+ y^2)<1.0001),
    Id=((x^2+(y+3)^2)<1.0001),
    Ie=((x-1)^2+ y^2)<=4),
    ud,u,uh,du;
real[int] z(3);
problem A(u,uh) = int2d(th) ((1+z[0]*Ib+z[1]*Ic+z[2]*Id) * (dx(u)*dx(uh)
    +dy(u)*dy(uh))) + on(aa,u=x^3-y^3);
z[0]=2; z[1]=3; z[2]=4;
A; ud=u;
ofstream f("J.txt");
func real J(real[int] & Z)
{
    for (int i=0;i<z.n;i++) z[i]=Z[i];
    A; real s= int2d(th) (Ie*(u-ud)^2);
    f<<s<<" "; return s;
}

real[int] dz(3), dJdz(3);

problem B(du,uh)
= int2d(th) ((1+z[0]*Ib+z[1]*Ic+z[2]*Id) * (dx(du)*dx(uh)+dy(du)*dy(uh)))
+ int2d(th) ((dz[0]*Ib+dz[1]*Ic+dz[2]*Id) * (dx(u)*dx(uh)+dy(u)*dy(uh)))
+ on(aa,du=0);

func real[int] DJ(real[int] &Z)
{
    for(int i=0;i<z.n;i++)
    { for(int j=0;j<dz.n;j++) dz[j]=0;
      dz[i]=1; B;
      dJdz[i]= 2*int2d(th) (Ie*(u-ud)*du);
    }
    return dJdz;
}

real[int] Z(3);
for(int j=0;j<z.n;j++) Z[j]=1;
BFGS(J,DJ,Z,eps=1.e-6,nbiter=15,nbiterline=20);
cout << "BFGS: J(z) = " << J(Z) << endl;
for(int j=0;j<z.n;j++) cout<<z[j]<<endl;
plot(ud,value=1,ps="u.eps");

```

In this example the sets B, C, D, E are circles of boundaries bb, cc, dd, ee are the domain Ω is the circle of boundary aa . The desired state u_d is the solution of the PDE for $b = 2, c = 3, d = 4$. The unknowns are packed into array z . Notice that it is necessary to recopy Z into z because one is a local variable while the other one is global. The program found $b = 2.00125, c = 3.00109, d = 4.00551$. Figure 3.12 shows u at convergence and the successive function evaluations of J . Note that an *adjoint state* could have been used. Define p by

$$-\nabla \cdot (\kappa \nabla p) = 2I_E(u - u_d), \quad p|_{\Gamma} = 0$$

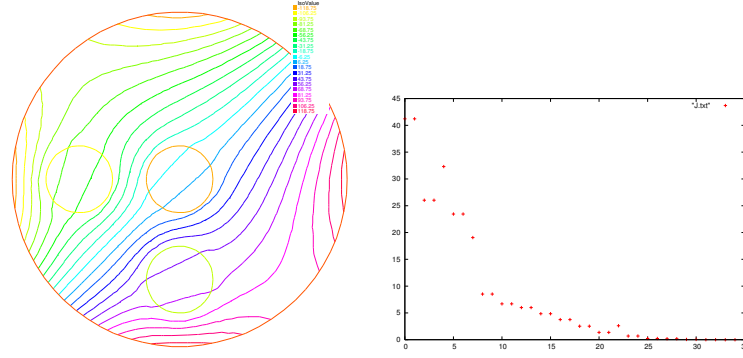


Figure 3.12: On the left the level lines of u . On the right the successive evaluations of J by BFGS (5 values above 500 have been removed for readability)

Consequently

$$\begin{aligned}\delta J &= - \int_{\Omega} (\nabla \cdot (\kappa \nabla p)) \delta u \\ &= \int_{\Omega} (\kappa \nabla p \cdot \nabla \delta u) = - \int_{\Omega} (\delta \kappa \nabla p \cdot \nabla u)\end{aligned}\quad (3.9)$$

Then the derivatives are found by setting $\delta b = 1, \delta c = \delta d = 0$ and so on:

$$J'_b = - \int_B \nabla p \cdot \nabla u, \quad J'_c = - \int_C \nabla p \cdot \nabla u, \quad J'_d = - \int_D \nabla p \cdot \nabla u$$

Remark As BFGS stores an $M \times M$ matrix where M is the number of unknowns, it is dangerously expensive to use this method when the unknown x is a Finite Element Function. One should use another optimizer such as the NonLinear Conjugate Gradient NLCG (also a key word of FreeFem++). See the file algo.edp in the examples folder.

3.14 A Flow with Shocks

Compressible Euler equations should be discretized with Finite Volumes or FEM with flux up-winding scheme but these are not implemented in FreeFem++ . Nevertheless acceptable results can be obtained with the method of characteristics provided that the mean values $\bar{f} = \frac{1}{2}(f^+ + f^-)$ are used at shocks in the scheme, and finally mesh adaptation .

$$\begin{aligned}\partial_t \rho + \bar{u} \nabla \rho + \bar{\rho} \nabla \cdot u &= 0 \\ \bar{\rho} (\partial_t u + \frac{\bar{\rho} u}{\bar{\rho}} \nabla u + \nabla p) &= 0 \\ \partial_t p + \bar{u} \nabla p + (\gamma - 1) \bar{p} \nabla \cdot u &= 0\end{aligned}\quad (3.10)$$

One possibility is to couple u, p and then update ρ , i.e.

$$\frac{1}{(\gamma - 1) \delta t \bar{p}^m} (p^{m+1} - p^m \circ X^m) + \nabla \cdot u^{m+1} = 0$$

$$\begin{aligned}
\frac{\bar{\rho}^m}{\delta t}(u^{m+1} - u^m \circ \tilde{X}^m) + \nabla p^{m+1} &= 0 \\
\rho^{m+1} &= \rho^m \circ X^m + \frac{\bar{\rho}^m}{(\gamma - 1)\bar{p}^m}(p^{m+1} - p^m \circ X^m)
\end{aligned} \tag{3.11}$$

A numerical result is given on Figure 3.13 and the FreeFem++ script is

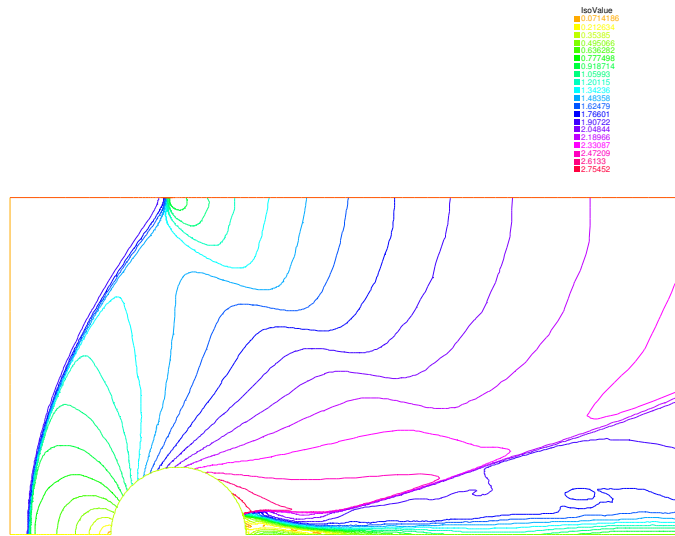


Figure 3.13: Pressure for a Euler flow around a disk at Mach 2 computed by (3.11)

```

verbosity=1;
int anew=1;
real x0=0.5,y0=0, rr=0.2;
border ccc(t=0,2){x=2-t;y=1;};
border ddd(t=0,1){x=0;y=1-t;};
border aaa1(t=0,x0-rr){x=t;y=0;};
border cercle(t=pi,0){ x=x0+rr*cos(t);y=y0+rr*sin(t);};
border aaa2(t=x0+rr,2){x=t;y=0;};
border bbb(t=0,1){x=2;y=t;};

int m=5; mesh Th;
if(anew) Th = buildmesh (ccc(5*m) + ddd(3*m) + aaa1(2*m) + cercle(5*m)
    + aaa2(5*m) + bbb(2*m) );
    else Th = readmesh ("Th_circle.mesh"); plot (Th,wait=0);

real dt=0.01, u0=2, err0=0.00625, pena=2;
fespace Wh(Th,P1);
fespace Vh(Th,P1);
Wh u,v,u1,v1,uh,vh;
Vh r,rh,r1;

```

```

macro dn(u) (N.x*dx(u)+N.y*dy(u) ) // def the normal derivative

if(anew){ u1= u0; v1= 0; r1 = 1;}
else {
    ifstream g("u.txt");g>>u1[];
    ifstream gg("v.txt");gg>>v1[];
    ifstream ggg("r.txt");ggg>>r1[];
    plot(u1,ps="eta.eps", value=1,wait=1);
    err0=err0/10; dt = dt/10;
}

problem eul(u,v,r,uh,vh,rh)
    = int2d(Th) ( (u*uh+v*vh+r*rh)/dt
                + ((dx(r)*uh+ dy(r)*vh) - (dx(rh)*u + dy(rh)*v))
                )
    + int2d(Th) (- (rh*convect([u1,v1],-dt,r1) + uh*convect([u1,v1],-dt,u1)
                + vh*convect([u1,v1],-dt,v1))/dt)
    + int1d(Th,6) (rh*u) // +int1d(Th,1) (rh*v)
    + on(2,r=0) + on(2,u=u0) + on(2,v=0);

int j=80;
for(int k=0;k<3;k++)
{
    if(k==20){ err0=err0/10; dt = dt/10; j=5;}
    for(int i=0;i<j;i++){
        eul; u1=u; v1=v; r1=abs(r);
        cout<<"k="<<k<<" E="<<int2d(Th) (u^2+v^2+r)<<endl;
        plot(r,wait=0,value=1);
    }
}
Th = adaptmesh (Th,r, nbvx=40000,err=err0,
    abseerror=1,nbjacoby=2, omega=1.8,ratio=1.8, nbsmooth=3,
    splitpbedge=1, maxsubdiv=5,rescaling=1) ;
plot(Th,wait=0);
u=u;v=v;r=r;

savemesh(Th,"Th_circle.mesh");
ofstream f("u.txt");f<<u[];
ofstream ff("v.txt");ff<<v[];
ofstream fff("r.txt");fff<<r[];
r1 = sqrt(u*u+v*v);
plot(r1,ps="mach.eps", value=1);
r1=r;
}

```

3.15 Classification of the equations

Summary *It is usually not easy to determine the type of a system. Yet the approximations and algorithms suited to the problem depend on its type:*

- *Finite Elements compatible (LBB conditions) for elliptic systems*
- *Finite difference on the parabolic variable and a time loop on each elliptic subsystem of parabolic systems; better stability diagrams when the schemes are implicit in time.*

- *Upwinding, Petrov-Galerkin, Characteristics-Galerkin, Discontinuous-Galerkin, Finite Volumes for hyperbolic systems plus, possibly, a time loop.*

When the system changes type, then expect difficulties (like shock discontinuities)!

Elliptic, parabolic and hyperbolic equations A partial differential equation (PDE) is a relation between a function of several variables and its derivatives.

$$F(\varphi(x), \frac{\partial \varphi}{\partial x_1}(x), \dots, \frac{\partial \varphi}{\partial x_d}(x), \frac{\partial^2 \varphi}{\partial x_1^2}(x), \dots, \frac{\partial^m \varphi}{\partial x_d^m}(x)) = 0 \quad \forall x \in \Omega \subset \mathcal{R}^d.$$

The range of x over which the equation is taken, here Ω , is called the *domain* of the PDE. The highest derivation index, here m , is called the *order*. If F and φ are vector valued functions, then the PDE is actually a *system* of PDEs.

Unless indicated otherwise, here by convention *one* PDE corresponds to one scalar valued F and φ . If F is linear with respect to its arguments, then the PDE is said to be *linear*.

The general form of a second order, linear scalar PDE is $\frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ and $A : B$ means $\sum_{i,j=1}^d a_{ij} b_{ij}$.

$$\alpha \varphi + a \cdot \nabla \varphi + B : \nabla(\nabla \varphi) = f \quad \text{in} \quad \Omega \subset \mathcal{R}^d,$$

where $f(x), \alpha(x) \in \mathcal{R}, a(x) \in \mathcal{R}^d, B(x) \in \mathcal{R}^{d \times d}$ are the PDE *coefficients*. If the coefficients are independent of x , the PDE is said to have *constant coefficients*.

To a PDE we associate a quadratic form, by replacing φ by 1, $\partial \varphi / \partial x_i$ by z_i and $\partial^2 \varphi / \partial x_i \partial x_j$ by $z_i z_j$, where z is a vector in \mathcal{R}^d :

$$\alpha + a \cdot z + z^T B z = f.$$

If it is the equation of an ellipse (ellipsoid if $d \geq 2$), the PDE is said to be *elliptic*; if it is the equation of a parabola or a hyperbola, the PDE is said to be *parabolic* or *hyperbolic*. If $A \equiv 0$, the degree is no longer 2 but 1, and for reasons that will appear more clearly later, the PDE is still said to be hyperbolic.

These concepts can be generalized to systems, by studying whether or not the polynomial system $P(z)$ associated with the PDE system has branches at infinity (ellipsoids have no branches at infinity, paraboloids have one, and hyperboloids have several).

If the PDE is not linear, it is said to be *non linear*. Those are said to be locally elliptic, parabolic, or hyperbolic according to the type of the linearized equation.

For example, for the non linear equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} = 1,$$

we have $d = 2, x_1 = t, x_2 = x$ and its linearized form is:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial \varphi}{\partial x} \frac{\partial^2 u}{\partial x^2} = 0,$$

which for the unknown u is locally elliptic if $\frac{\partial \varphi}{\partial x} < 0$ and locally hyperbolic if $\frac{\partial \varphi}{\partial x} > 0$.

Examples Laplace's equation is elliptic:

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \cdots + \frac{\partial^2\varphi}{\partial x_d^2} = f, \quad \forall x \in \Omega \subset \mathcal{R}^d.$$

The *heat* equation is parabolic in $Q = \Omega \times]0, T[\subset \mathcal{R}^{d+1}$:

$$\frac{\partial\varphi}{\partial t} - \mu\Delta\varphi = f \quad \forall x \in \Omega \subset \mathcal{R}^d, \quad \forall t \in]0, T[.$$

If $\mu > 0$, the *wave* equation is hyperbolic:

$$\frac{\partial^2\varphi}{\partial t^2} - \mu\Delta\varphi = f \quad \text{in } Q.$$

The *convection diffusion* equation is parabolic if $\mu \neq 0$ and hyperbolic otherwise:

$$\frac{\partial\varphi}{\partial t} + a\nabla\varphi - \mu\Delta\varphi = f.$$

The *biharmonic* equation is elliptic:

$$\Delta(\Delta\varphi) = f \quad \text{in } \Omega.$$

Boundary conditions A relation between a function and its derivatives is not sufficient to define the function. Additional information on the boundary $\Gamma = \partial\Omega$ of Ω , or on part of Γ is necessary.

Such information is called a *boundary condition*. For example,

$$\varphi(x) \text{ given, } \forall x \in \Gamma,$$

is called a *Dirichlet boundary condition*. The *Neumann* condition is

$$\frac{\partial\varphi}{\partial n}(x) \text{ given on } \Gamma \quad (\text{or } n \cdot B\nabla\varphi, \text{ given on } \Gamma \text{ for a general second order PDE})$$

where n is the normal at $x \in \Gamma$ directed towards the exterior of Ω (by definition $\frac{\partial\varphi}{\partial n} = \nabla\varphi \cdot n$). Another classical condition, called a *Robin* (or *Fourier*) condition is written as:

$$\varphi(x) + \beta(x) \frac{\partial\varphi}{\partial n}(x) \text{ given on } \Gamma.$$

Finding a set of boundary conditions that defines a unique φ is a difficult art.

In general, an elliptic equation is well posed (*i.e.* φ is unique) with one Dirichlet, Neumann or Robin conditions on the whole boundary.

Thus, Laplace's equations is well posed with a Dirichlet or Neumann condition but also with

$$\varphi \text{ given on } \Gamma_1, \quad \frac{\partial\varphi}{\partial n} \text{ given on } \Gamma_2, \quad \Gamma_1 \cup \Gamma_2 = \Gamma, \quad \Gamma_1 \cap \Gamma_2 = \emptyset.$$

Parabolic and hyperbolic equations rarely require boundary conditions on all of $\Gamma \times]0, T[$. For instance, the heat equation is well posed with

$$\varphi \text{ given at } t = 0 \text{ and Dirichlet or Neumann or mixed conditions on } \partial\Omega.$$

Here t is time so the first condition is called an initial condition. The whole set of conditions are also called Cauchy conditions.

The wave equation is well posed with

$$\varphi \text{ and } \frac{\partial\varphi}{\partial t} \text{ given at } t = 0 \text{ and Dirichlet or Neumann or mixed conditions on } \partial\Omega.$$

Chapter 4

Syntax

4.1 Data Types

In essence FreeFem++ is a compiler: its language is typed, polymorphic, with exception and reentrant. Every variable must be declared of a certain type, in a declarative statement; each statement are separated from the next by a semicolon ‘;’. The language allows the manipulation of basic types integers (`int`), reals (`real`), strings (`string`), arrays (example: `real[int]`), bidimensional (2D) finite element meshes (`mesh`), 2D finite element spaces (`fespace`), analytical functions (`func`), arrays of finite element functions (`func[basic-type]`), linear and bilinear operators, sparse matrices, vectors, etc. For instance

```
int i,n=20;                                // i,n are integer.
real[int] xx(n),yy(n);                     // two array of size n
for (i=0;i<=20;i++)                        // which can be used in statements such as
{ xx[i]= cos(i*pi/10); yy[i]= sin(i*pi/10); }
```

The life of a variable is the current block {...}, except the `fespace` variable, and the variables local to a block are destroyed at the end of the block as follows.

Example 4.1

```
real r= 0.01;
mesh Th=square(10,10);                    // unit square mesh
fespace Vh(Th,P1);                        // P1 lagrange finite element space
Vh u = x+ exp(y);
func f = z * x + r * log(y);
plot(u,wait=true);
{
    real r = 2;                            // new block
    fespace Vh(Th,P1);                     // not the same r
}                                           // error because Vh is a global name
                                           // end of block
                                           // here r back to 0.01
```

The type declarations are compulsory in FreeFem++ ; in the end this feature is an asset because it is easy to make bugs in a language with many implicit types. The variable name is just an alphanumeric string, the underscore character “_” is not allowed, because it will be used as an operator in the future.

4.2 List of major types

bool is used for logical expression and flow-control. The result of a comparison is a boolean type as in

```
bool fool=(1<2);
```

which makes fool to be true. Similar examples can be built with `==`, `<=`, `>=`, `<`, `>`, `!=`.

int declares an integer (i.e. long in C++).

string declare the variable to store a text enclosed within double quotes, such as:

```
"This is a string in double quotes."
```

real declares the variable to store a number such as "12.345". (i.e. double in C++).

complex Complex numbers, such as $1 + 2i$, FreeFem++ understand that $i = \sqrt{-1}$ (i.e. `complex<double>` in C++).

```
complex a = 1i, b = 2 + 3i;
cout << "a + b = " << a + b << endl;
cout << "a - b = " << a - b << endl;
cout << "a * b = " << a * b << endl;
cout << "a / b = " << a / b << endl;
```

Here's the output;

```
a + b = (2,4)
a - b = (-2,-2)
a * b = (-3,2)
a / b = (0.230769,0.153846)
```

ofstream to declare an output file .

ifstream to declare an input file .

real[int] declares a variable that stores multiple real numbers with integer index.

```
real[int] a(5);
a[0] = 1; a[1] = 2; a[2] = 3.333333; a[3] = 4; a[4] = 5;
cout << "a = " << a << endl;
```

This produces the output;

```
a = 5      :
    1      2      3.33333  4      5
```

real[string] declares a variable that store multiple real numbers with string index.

string[string] declares a variable that store multiple strings with string index.

func defines a function without argument, if independent variables are x , y . For example

```
func f=cos(x)+sin(y) ;
```

Remark that the function's type is given by the expression's type. Raising functions to a numerical power is done, for instance, by x^1 , $y^{0.23}$.

mesh creates the triangulation, see Section 5.

fespace defines a new type of finite element space, see Section Section 6.

problem declares the weak form of a partial differential problem without solving it.

solve declares a problem and solves it.

varf defines a full variational form.

matrix defines a sparse matrix.

4.3 Global Variables

The names $x, y, z, label, region, P, N, nu_triangle \dots$ are reserved words used to link the language to the finite element tools:

x is the x coordinate of the current point (real value)

y is the y coordinate of the current point (real value)

z is the z coordinate of the current point (real value)

label contains the label number of a boundary if the current point is on a boundary, 0 otherwise (int value).

region returns the region number of the current point (x,y) (int value).

P gives the current point (\mathbb{R}^2 value.). By $P.x$, $P.y$, we can get the x , y components of P . Also $P.z$ is reserved and can be used in 3D.

N gives the outward unit normal vector at the current point if it is on a curve defined by **border** (\mathbb{R}^3 value). $N.x$ and $N.y$ are x and y components of the normal vector. $N.z$ is reserved. .

lenEdge gives the length of the current edge

$$\text{lenEdge} = |q^i - q^j| \quad \text{if the current edge is } [q^i, q^j]$$

hTriangle gives the size of the current triangle

nuTriangle gives the index of the current triangle (int value).

nuEdge gives the index of the current edge in the triangle (int value).

nTonEdge gives the number of adjacent triangle of the current edge (integer).

area give the area of the current triangle (real value).

volume give the volume of the current tetrahedra (real value).

cout is the standard output device (default is console). On MS-Windows, the standard output is only the console, at this time. `ostream`

cin is the standard input device (default is keyboard). (`istreamvalue`).

endl adds an "end of line" to the input/output flow.

true means "true" in `bool` value.

false means "false" in `bool` value.

pi is the `realvalue` approximation value of π .

4.4 System Commands

Here is how to show all the types, and all the operator and functions of a `FreeFem++` program:

```
dumptable(cout);
```

To execute a system command in the string (not implemented on Carbon MacOS), which return the value of the system call.

```
system("shell command"); // after version 3.12-1
exec("shell command");
```

This is useful to launch another executable from within `FreeFem++` . On MS-Windows, the full path of the executable. For example, if there is the command "ls.exe" in the subdirectory "c:\cygwin\bin\", then we must write

```
exec("c:\\cygwin\\bin\\ls.exe");
```

Another useful system command is `assert()` to make sure something is true.

```
assert(version>=1.40);
```

4.5 Arithmetics

On integers , $+$, $-$, $*$ are the usual arithmetic summation (plus), subtraction (minus) and multiplication (times), respectively. The operators $/$ and $\%$ yield the quotient and the remainder from the division of the first expression by the second. If the second number of $/$ or $\%$ is zero the behavior is undefined. The *maximum* or *minimum* of two integers a , b are obtained by $\max(a, b)$ or $\min(a, b)$. The power a^b of two integers a , b is calculated by writing a^b . The classical C++ "arithmetical if" expression $a ? b : c$ is equal to the value of expression b if the value of expression a is true otherwise is equal to value of expression c .

Example 4.2 Computations with the integers

```
int a = 12, b = 5;
cout << "plus, minus of "<<a<< " and "<<b<< " are "<<a+b<< " , "<<a-b<< endl;
cout << "multiplication, quotient of them are "<<a*b<< " , "<<a/b<< endl;
cout << "remainder from division of "<<a<< " by "<<b<< " is "<<a%b<< endl;
cout << "the minus of "<<a<< " is "<< -a << endl;
cout <<a<< " plus -"<<b<< " need bracket: "<<a<< " + (-"<<b<< " ) = "<<a + (-b) << endl;
cout << "max and min of "<<a<< " and "<<b<< " is "<<max(a,b)<< " , "<<min(a,b)<< endl;
cout <<b<< "th power of "<<a<< " is "<<a^b<< endl;
cout << " min == (a < b ? a : b ) is " << (a < b ? a : b) << endl;

b=0;
cout <<a<< "/0"<< " is "<< a/b << endl;
cout <<a<< "%0"<< " is "<< a%b << endl;
```

produce the following result:

```
plus, minus of 12 and 5 are 17, 7
multiplication, quotient of them are 60, 2
remainder from division of 12 by 5 is 2
the minus of 12 is -12
12 plus -5 need bracket :12+(-5)=7
max and min of 12 and 5 is 12,5
5th power of 12 is 248832
min == (a < b ? a : b) is 5
12/0 : long long long
Fatal error : ExecError Div by 0 at exec line 9
Exec error : exit
```

By the relation $integer \subset real$, the operators “ $+$, $-$, $*$, $/$, $\%$ ” and “**max**, **min**, $^$ ” are extended to real numbers or variables. However, $\%$ calculates the remainder of the integer parts of two real numbers.

The following are examples similar to Example 4.2

```
real a=sqrt(2.), b = pi;
cout << "plus, minus of "<<a<< " and "<<pi<< " are "<< a+b << " , "<< a-b << endl;
cout << "multiplication, quotient of them are "<<a*b<< " , "<<a/b<< endl;
cout << "remainder from division of "<<a<< " by "<<b<< " is "<< a%b << endl;
cout << "the minus of "<<a<< " is "<< -a << endl;
cout <<a<< " plus -"<<b<< " need bracket : "<<a<< " + (-"<<b<< " ) = "<<a + (-b) << endl;
```

It gives the following output:

```
plus, minus of 1.41421 and 3.14159 are 4.55581, -1.72738
multiplication, quotient of them are 4.44288, 0.450158
remainder from division of 1.41421 by 3.14159 is 1
the minus of 1.41421 is -1.41421
1.41421 plus -3.14159 need bracket :1.41421+(-3.14159)=-1.72738
```

By the relation

$$\text{bool} \subset \text{int} \subset \text{real} \subset \text{complex},$$

the operators “+”, “-”, “*”, “/” and “^” are also applicable on complex-typed variables, but “%”, “max”, “min” cannot be used. Complex numbers such as $5+9i$, $i = \sqrt{-1}$, are valid expressions. With real variables $a=2.45$, $b=5.33$, complex numbers like $a+ib$ and $a+i\sqrt{2.0}$ must be declared by

```
complex z1 = a+b*1i, z2=a+sqrt(2.0)*1i;
```

The imaginary and real parts of a complex number z can be obtained with **imag** and **real**.

The conjugate of $a+bi$ (a, b are reals) is defined by $a-bi$, which can also be computed with the operator “conj”, by **conj**($a+b*1i$) in FreeFem++ .

Internally the complex number $z = a+ib$ is considered as the pair (a, b) of real numbers a, b . We can attach to it the point (a, b) in the Cartesian plane where the x -axis is for the real part and the y -axis for the imaginary part. The same point (a, b) has a representation with polar coordinate (r, ϕ) , So z is also $z = r(\cos \phi + i \sin \phi)$, $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$; r is called the *modulus* and ϕ the *argument* of z . In the following example, we shall show them using FreeFem++ programming, and *de Moivre’s formula* $z^n = r^n(\cos n\phi + i \sin n\phi)$.

Example 4.3

```
real a=2.45, b=5.33;
complex z1=a+b*1i, z2 = a+sqrt(2.)*1i;
func string pc(complex z) // printout complex to (real)+i(imaginary)
{
    string r = "("+real(z);
    if (imag(z)>=0) r = r+"";
    return r+imag(z)+"i)";
}

// printout complex to |z|*(cos(arg(z))+i*sin(arg(z)))
func string toPolar(complex z)
{
    return abs(z)+"*(cos("+arg(z)+")+i*sin("+arg(z)+"))";
}

cout <<"Standard output of the complex "<<pc(z1)<<" is the pair "
    <<z1<<endl;
cout <<"Plus, minus of "<<pc(z1)<<" and "<<pc(z2)<<" are "<< pc(z1+z2)
    <<","<< pc(z1-z2) << endl;
cout <<"Multiplication, quotient of them are "<<pc(z1*z2)<<","<< "
    <<pc(z1/z2)<< endl;
cout <<"Real/imaginary part of "<<pc(z1)<<" is "<<real(z1)<<","<< "
    <<imag(z1)<<endl;
cout <<"Absolute of "<<pc(z1)<<" is "<<abs(z1)<<endl;
cout <<pc(z2)<<" = "<<toPolar(z2)<<endl;
cout <<" and polar("<<abs(z2)<<","<<arg(z2)<<") = "
```



```

    << pc(polar(abs(z2),arg(z2)))<<endl;
cout <<"de Moivre's formula: "<<pc(z2)<<"^3 = "<<toPolar(z2^3)<<endl;
cout <<"conjugate of "<<pc(z2)<<" is "<<pc(conj(z2))<<endl;
cout <<pc(z1)<<"^"<<pc(z2)<<" is "<< pc(z1^z2) << endl;

```

Here's the output from Example 4.3

```

Standard output of the complex (2.45+5.33i) is the pair (2.45,5.33)
Plus, minus of (2.45+5.33i) and (2.45+1.41421i) are (4.9+6.74421i), (0+3.91579i)
Multiplication, quotient of them are (-1.53526+16.5233i), (1.692+1.19883i)
Real/imaginary part of (2.45+5.33i) is 2.45, 5.33
Absolute of (2.45+5.33i) is 5.86612
(2.45+1.41421i) = 2.82887*(cos(0.523509)+i*sin(0.523509))
    and polar(2.82887,0.523509) = (2.45+1.41421i)
de Moivre's formula: (2.45+1.41421i)^3
                      = 22.638*(cos(1.57053)+i*sin(1.57053))
conjugate of (2.45+1.41421i) is (2.45-1.41421i)
(2.45+5.33i)^(2.45+1.41421i) is (8.37072-12.7078i)

```

4.6 string expression

In the following example you some example string expression

```

string tt="toto1"+1+" -- 77";           // string concatenation
string t1="0123456789";
string t2;                               // new operator

t2 ="12340005678";
t2(4:3) = "abcdefghijk-";
string t55=t2(4:3);

// t2 = "12340abcdefghijk-005678";

cout << t2 << endl;
cout << " find abc " << t2.find("abc") << endl;
cout << "r find abc " << t2.rfind("abc") << endl;
cout << " find abc from 10 " << t2.find("abc",10) << endl;
cout << " ffind abc from 10 " <<t2.rfind("abc",10) << endl;
cout << " " << string("abcc").length << endl;
cout << " t55 " << t55 << endl;
{
// add getline version 3.0-6 jan 2009 FH
string s;
ifstream toto("xyf");
for (int i=0;i<10;++i)
{
    getline(toto,s);
    cout << i << " : " << s << endl;
}
}

```

4.7 Functions of one Variable

Fundamental functions are built into FreeFem++ as well as The *power function* x^y = `pow(x,y)` = x^y ; the *exponent function* `exp(x)` ($= e^x$), the *logarithmic function* `log(x)` ($= \ln x$) or `log10(x)` ($= \log_{10} x$); the *trigonometric functions* `sin(x)`, `cos(x)`, `tan(x)` assume angles measured in *radians*; the inverse of $\sin x$, $\cos x$, $\tan x$ (called *circular function* or *inverse trigonometric function*) `asin(x)` ($= \arcsin x$), `acos(x)` ($= \arccos x$), `atan(x)` ($= \arctan x$) are also implemented; the `atan2(x,y)` function computes the principal value of the arc tangent of y/x , using the signs of both arguments to determine the quadrant of the return value; the *hyperbolic functions*,

$$\sinh x = (e^x - e^{-x})/2, \quad \cosh x = (e^x + e^{-x})/2.$$

and $\tanh x = \sinh x / \cosh x$ called by `sinh(x)`, `cosh(x)`, `tanh(x)`, `asinh(x)`, `acosh(x)` and `atanh(x)`.

$$\sinh^{-1} x = \ln \left[x + \sqrt{x^2 + 1} \right], \quad \cosh^{-1} x = \ln \left[x + \sqrt{x^2 - 1} \right].$$

The real function which rounds a real to an integer `floor(x)` rounds to largest integral value not greater than x , `ceil(x)` round to smallest integral value not less than x ; similarly `rint(x)` returns the integral value nearest to x (according to the prevailing rounding mode) in floating-point format)..

Elementary Functions denotes for us the class of functions presented above (polynomials, exponential, logarithmic, trigonometric, circular) and the functions obtained from those by the four arithmetic operations

$$f(x) + g(x), f(x) - g(x), f(x)g(x), f(x)/g(x)$$

and by composition $f(g(x))$, each applied a finite number of times. In FreeFem++ , all elementary functions can thus be created. The derivative of an elementary function is also an elementary function; however, the indefinite integral of an elementary function cannot always be expressed in terms of elementary functions.

Example 4.4 *The following is an example where an elementary function is used to build the border of a domain. Cardioid*

```
real b = 1.;
real a = b;
func real phix(real t)
{
    return (a+b)*cos(t)-b*cos(t*(a+b)/b);
}
func real phiy(real t)
{
    return (a+b)*sin(t)-b*sin(t*(a+b)/b);
}
border C(t=0,2*pi) { x=phix(t); y=phiy(t); }
mesh Th = buildmesh(C(50));
```

Taking the principal value, we can define $\log z$ for $z \neq 0$ by

$$\ln z = \ln |z| + i \arg z.$$

Using FreeFem++ , we calculated `exp(1+4i)`, `sin(pi+1i)`, `cos(pi/2-1i)` and `log(1+2i)`, we then have

$$\begin{aligned} & -1.77679 - 2.0572i, \quad 1.8896710^{-16} - 1.1752i, \\ & 9.4483310^{-17} + 1.1752i, \quad 0.804719 + 1.10715i. \end{aligned}$$

Random Functions can be define as FreeFem++ has a Mersenne Twister function (see page <http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html> for full detail). It is a very fast and accurate random number generator Of period $2^{219937}-1$, and the functions which calls it are:

- `randint32()` generates unsigned 32-bit integers.
- `randint31()` generates unsigned 31-bit integers.
- `randreal1()` generates uniform real in $[0, 1]$ (32-bit resolution).
- `randreal2()` generates uniform real in $[0, 1)$ (32-bit resolution).
- `randreal3()` generates uniform real in $(0, 1)$ (32-bit resolution).
- `randres53()` generates uniform real in $[0, 1)$ with 53-bit resolution.
- `randinit(seed)` initializes the state vector by using one 32-bit integer "seed", which may be zero.

Library Functions form the mathematical library (version 2.17).

- the functions `j0(x)`, `j1(x)`, `jn(n,x)`, `y0(x)`, `y1(x)`, `yn(n,x)` are the Bessel functions of first and second kind.

The functions `j0(x)` and `j1(x)` compute the Bessel function of the first kind of the order 0 and the order 1, respectively; the function `jn(n, x)` computes the Bessel function of the first kind of the integer order n .

The functions `y0(x)` and `y1(x)` compute the linearly independent Bessel function of the second kind of the order 0 and the order 1, respectively, for the positive integer value x (expressed as a real); the function `yn(n, x)` computes the Bessel function of the second kind for the integer order n for the positive integer value x (expressed as a real).

- the function `tgamma(x)` calculates the Γ function of x . `lgamma(x)` calculates the natural logarithm of the absolute value of the Γ function of x .
- The `erf(x)` function calculates the error function, where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$. The `erfc(x)` = function calculates the complementary error function of x , i.e. $\text{erfc}(x) = 1 - \text{erf}(x)$.

4.8 Functions of two Variables

4.8.1 Formula

The general form of real functions of two independent variables a, b is usually written as $c = f(a, b)$. In FreeFem++ , x , y and z are reserved word as explained in in Section 4.3. So when the two variables of the function are x and y , we may define the function without its argument, for example

```
func f=cos(x)+sin(y) ;
```

Remark that the function type is given by the expression type. The power operator can be used in functions such as x^1 , $y^{0.23}$. In `func`, we can write an elementary function as follows

```
func f = sin(x)*cos(y);
func g = (x^2+3*y^2)*exp(1-x^2-y^2);
func h = max(-0.5, 0.1*log(f^2+g^2));
```

Complex valued function create functions with 2 variables x, y as follows,

```
mesh Th=square(20,20,[-pi+2*pi*x,-pi+2*pi*y]); //      ] - pi, pi[2
fespace Vh(Th,P2);
func z=x+y*I; //      z = x + iy
func f=imag(sqrt(z)); //      f =  $\Im\sqrt{z}$ 
func g=abs( sin(z/10)*exp(z^2/10) ); //      g = |sin z/10 exp z2/10|
Vh fh = f; plot(fh); //      contour lines of f
Vh gh = g; plot(gh); //      contour lines of g
```

We call also construct *elementary functions of two variables* from elementary functions $f(x)$ or $g(y)$ by the four arithmetic operations plus composition applied a finite number of times.

4.8.2 FE-functions

Finite element functions are also constructed like elementary functions by an arithmetic formula involving elementary functions. The difference is that they are evaluated at declaration time and FreeFem++ stores the array of its values at the places associated with the degree of freedom of the finite element type. By opposition elementary functions are evaluated only when needed. Hence FE-functions are not defined only by their formula but also by the mesh and the finite element which enter in their definitions. If the value of a FE-function is requested at a point which is not a degree of freedom, an interpolation is used, leading to an interpolation error, while by contrast, an elementary function can be evaluated at any point exactly.

```
func f=x^2*(1+y)^3+y^2;
mesh Th = square(20,20,[-2+4*x,-2+4*y]); //      square ] - 2, 2[2
fespace Vh(Th,P1);
Vh fh=f; //      fh is the projection of f to Vh (real value)
func zf=(x^2*(1+y)^3+y^2)*exp(x+I*y);
Vh<complex> zh = zf; //      zh is the projection of zf
//      to complex value Vh space
```

The construction of $fh (=f_h)$ is explained in Section 6.

Note 4.1 The command `plot` applies only for real or complex FE-functions (2d or 3d) and not to elementary functions.

Complex valued functions create functions with 2 variables x, y as follows,

```

mesh Th=square(20,20, [-pi+2*pi*x, -pi+2*pi*y]);           //      ] -  $\pi, \pi$ 
fespace Vh(Th,P2);
func z=x+y*1i;                                           //       $z = x + iy$ 
func f=imag(sqrt(z));                                     //       $f = \Im\sqrt{z}$ 
func g=abs( sin(z/10)*exp(z^2/10) );                     //       $g = |\sin z/10 \exp z^2/10|$ 
Vh fh = f; plot(fh);                                     //      Fig. 4.1 isovalue of  $f$ 
Vh gh = g; plot(gh);                                     //      Fig. 4.2 isovalue of  $g$ 

```

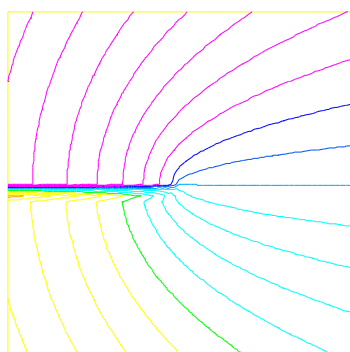


Figure 4.1: $\Im\sqrt{z}$ has branch

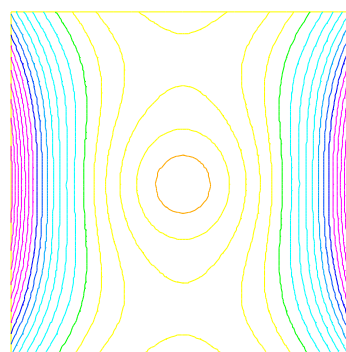


Figure 4.2: $|\sin(z/10) \exp(z^2/10)|$

4.9 Arrays

An *array* stores multiple objects, and there are 2 kinds of arrays: The first is similar to *vector*, i.e. arrays with *integer indices* and the second type is arrays with *string indices*. In the first case, the size of the array must be known at execution time, and implementation is done with the `KN<>` class and all the vector operator of `KN<>` are implemented. For instance

```

real [int] tab(10), tab1(10);                               //      2 array of 10 real
real [int] tab2;                                             //      bug array with no size
tab = 1.03;                                                  //      set all the array to 1.03
tab[1]=2.15;
cout << tab[1] << " " << tab[9] << " size of tab = "
      << tab.n << " min: " << tab.min << " max:" << tab.max
      << " sum : " << tab.sum << endl;                       //
tab.resize(12);                                              //      change the size of array tab
                                                    //      to 12 with preserving first value
                                                    //      set unset value
tab(10:11)=3.14;
cout << " resize tab: " << tab << endl;
real [string] tt;
tt["+"]=1.5;

```

```

cout<<tt["a"]<<" " <<tt["+"]<<endl;
real[int] a(5),b(5),c(5),d(5);
a = 1;
b = 2;
c = 3;
a[2]=0;
d = ( a ? b : c ); // for i = 0, n-1 : d[i] = a[i] ? b[i] : c[i] ,
cout << " d = ( a ? b : c ) is " << d << endl;
d = ( a ? 1 : c ); // for i = 0, n-1: d[i] = a[i] ? 1 : c[i] , (v2.23-1)
d = ( a ? b : 0 ); // for i = 0, n-1: d[i] = a[i] ? b[i] : 0 , (v2.23-1)
d = ( a ? 1 : 0 ); // for i = 0, n-1: d[i] = a[i] ? 0 : 1 , (v2.23-1)
tab.sort ; // sort the array tab (version 2.18)
cout << " tab (after sort) " << tab << endl;
int[int] ii(0:d.n-1); // set array ii to 0,1, ..., d.n-1 (v3.2)
d=-1:-5; // set d to -1,-2, .. -5 (v3.2)

sort(d,ii); // sort array d and ii in parallel
cout << " d " << d << "\n ii = " << ii << endl;

```

produces the output

```

2.15 1.03 size of tab = 10 min: 1.03 max:2.15 sum : 11.42
resize tab: 12
    1.03    2.15    1.03    1.03    1.03
    1.03    1.03    1.03    1.03    1.03
    3.14    3.14
0  1.5
d = ( a ? b : c ) is 5
    3    3    2    3    3
tab (after sort) 12
1.03 1.03 1.03 1.03 1.03
1.03 1.03 1.03 1.03 2.15
3.14 3.14
d 5
-5 -4 -3 -2 -1

ii = 5
 4  3  2  1  0

```

Arrays can be set like in matlab or scilab with the operator `::`, the array generator of `a:c` is equivalent to `a:1:c`, and the array set by `a:b:c` is set to size $\lfloor (b-a)/c \rfloor + 1$ and the value i is set by $a + i(b-a)/c$.

There are `int`, `real`, `complex` arrays with, in the third case, two operators (`.in`, `.re`) to generate the real and imaginary real array from the complex array (without copy) :

```

// version 3.2 mai 2009
// like matlab. and scilab

{
int[int] tt(2:10); // 2,3,4,5,6,7,8,9,10
int[int] t1(2:3:10); // 2,5,8,
cout << " tt(2:10)= " << tt << endl;
cout << " t1(2:3:10)= " << t1 << endl;
tt=1:2:5;
cout << " 1.:2:5 => " << tt << endl;
}

```

```

{
real[int] tt(2:10);                                // 2,3,4,5,6,7,8,9,10
real[int] t1(2.:3:10.);                             // 2,5,8,
cout << " tt(2:10)= " << tt << endl;
cout << " t1(2:3:10)= " << t1 << endl;
tt=1.:0.5:3.999;
cout << " 1.:0.5:3.999 => " << tt << endl;
}
{
complex[int] tt(2.+0i:10.+0i);                      // 2,3,4,5,6,7,8,9,10
complex[int] t1(2.:3.:10.);                          // 2,5,8,
cout << " tt(2.+0i:10.+0i)= " << tt << endl;
cout << " t1(2.:3.:10.)= " << t1 << endl;
cout << " tt.re real part array " << tt.re << endl ;
// the real part array of the complex array
cout << " tt.im imag part array " << tt.im << endl ;
// the imag part array of the complex array

}

```

The output is :

```

tt(2:10)= 9
 2  3  4  5  6
 7  8  9 10
t1(2:3:10)= 3
 2  5  8
1.:2:5 => 3
 1  3  5
tt(2:10) == 9
 2  3  4  5  6
 7  8  9 10
t1(2.:3:10.)= 3
 2  5  8
1.:0.5:3.999 => 6
 1 1.5  2 2.5  3
3.5
tt(2.+0i:10.+0i)= 9
(2,0) (3,0) (4,0) (5,0) (6,0)
(7,0) (8,0) (9,0) (10,0)
t1(2.:3.:10.);= 3
(2,0) (5,0) (8,0)

tt.re real part array  9
 2  3  4  5  6
 7  8  9 10
tt.im imag part array  9
 0  0  0  0  0
 0  0  0  0
2

```

the all integer array operators are :

```

{

```

```

int N=5;
real[int] a(N),b(N),c(N);
a =1;
a(0:4:2) = 2;
a(3:4) = 4;
cout <<" a = " << a << endl;
b = a+ a;
cout <<" b = a+a : " << b << endl;
b += a;
cout <<" b += a : " << b << endl;
b += 2*a;
cout <<" b += 2*a : " << b << endl;
b /= 2;
cout <<" b /= 2 : " << b << endl;
b .*= a; // same b = b .* a
cout << "b.*=a; b =" << b << endl;
b ./= a; // same b = b ./ a
cout << "b./=a; b =" << b << endl;
c = a+b;
cout << " c =a+b : c=" << c << endl;
c = 2*a+4*b;
cout << " c =2*a+4b : c= " << c << endl;
c = a+4*b;
cout << " c =a+4b : c= " << c << endl;
c = -a+4*b;
cout << " c =-a+4b : c= " << c << endl;
c = -a-4*b;
cout << " c =-a-4b : c= " << c << endl;
c = -a-b;
cout << " c =-a-b : c= " << c << endl;

c = a .* b;
cout << " c =a.*b : c= " << c << endl;
c = a ./ b;
cout << " c =a./b : c= " << c << endl;
c = 2 * b;
cout << " c =2*b : c= " << c << endl;
c = b*2 ;
cout << " c =b*2 : c= " << c << endl;

/* this operator do not exist
c = b/2 ;
cout << " c =b/2 : c= " << c << endl;
*/

// ---- the methods --
cout << " ||a||_1 = " << a.l1 << endl; //
cout << " ||a||_2 = " << a.l2 << endl; //
cout << " ||a||_inf = " << a.linfty << endl; //
cout << " sum a_i = " << a.sum << endl; //
cout << " max a_i = " << a.max << " a[ " << a.imax << " ] = " << a[a.imax] << endl; //
cout << " min a_i = " << a.min << " a[ " << a.imin << " ] = " << a[a.imin] << endl; //

cout << " a'*a = " << (a'*a) << endl; //

```



```

cout << " a quantile 0.2 = " << a.quantile(0.2) << endl; // array mapping
int[int] I=[2,3,4,-1,3];
b=c=-3;
b= a(I); // for( i=0;i<b.n;i++) if(I[i] >=0) b[i]=a[I[i]];
c(I)= a; // for( i=0;i<I.n;i++) if(I[i] >=0) C(I[i])=a[i];
cout << " b = a(I) : " << b << "\n c(I) = a " << c << endl;
c(I) += a; // for( i=0;i<I.n;i++) if(I[i] >=0) C(I[i])+=a[i];
cout << " b = a(I) : " << b << "\n c(I) = a " << c << endl;

}

```

produce the output

```

5
      3      3      2      3      3
==  3      3      2      3      3
a = 5
      2      1      2      4      4
b = a+a : 5
      4      2      4      8      8
b += a : 5
      6      3      6     12     12
b += 2*a : 5
     10      5     10     20     20
b /= 2 : 5
      5     2.5      5     10     10
b.*=a; b =5
     10     2.5     10     40     40
b./=a; b =5
      5     2.5      5     10     10
c =a+b : c=5
      7     3.5      7     14     14
c =2*a+4b : c= 5
     24     12     24     48     48
c =a+4b : c= 5
     22     11     22     44     44
c =-a+4b : c= 5
     18      9     18     36     36
c =-a-4b : c= 5
    -22    -11    -22    -44    -44
c =-a-b : c= 5
     -7    -3.5     -7    -14    -14

```

```

c =a.*b   : c= 5
           10      2.5      10      40      40

c =a./b   : c= 5
           0.4      0.4      0.4      0.4      0.4

c =2*b    : c= 5
           10      5       10      20      20

c =b*2    : c= 5
           10      5       10      20      20

||a||_1    = 13
||a||_2    = 6.403124237
||a||_infty = 4
sum a_i    = 13
max a_i    = 4 a[ 3 ] = 4
min a_i    = 1 a[ 1 ] = 1
a'*a      = 41
a quantile 0.2 = 2
b = a(I)   : 5
           2   4   4  -3   4

c(I) = a 5
-3 -3  2  4  2

b = a(I)   : 5
           2   4   4  -3   4

c(I) = a 5
-3 -3  4  9  4

```

Note 4.2 *Quantiles are points taken at regular intervals from the cumulative distribution function of a random variable. Here the array values are random.*

This statistical function `a.quantile(q)` computes v from an array a of size n for a given number $q \in]0, 1[$ such that

$$\#\{i/a[i] < v\} \sim q * n$$

*; it is equivalent to $v = a[q * n]$ when the array a is sorted.*

Example of array with renumbering (version 2.3 or better) . The renumbering is always given by an integer array, and if a value in the array is negative, the mapping is not imaged, so the value is not set.

```

int[int] I=[2,3,4,-1,0];           // the integer mapping to set the renumbering
b=c=-3;
b= a(I);                          // for( i=0;i<b.n;i++) if(I[i] >=0) b[i]=a[I[i]];
c(I)= a;                          // for( i=0;i<I.n;i++) if(I[i] >=0) C(I[i])=a[i];
cout << " b = a(I) : " << b << "\n c(I) = a " << c << endl;

```

The output is

```

b = a(I) : 5
           2         4         4        -3         2

c(I) = a 5
          4        -3         2         1         2

```

4.9.1 Arrays with two integer indices versus matrices

Some example are given below to transform full matrices into sparse matrices.

```

int N=3,M=4;

real[int,int] A(N,M);
real[int] b(N),c(M);
b=[1,2,3];
c=[4,5,6,7];

complex[int,int] C(N,M);
complex[int] cb=[1,2,3],cc=[10i,20i,30i,40i];

b=[1,2,3];

int [int] I=[2,0,1];
int [int] J=[2,0,1,3];

A=1; // set the all matrix
A(2,:) = 4; // the full line 2
A(:,1) = 5; // the full column 1
A(0:N-1,2) = 2; // set the column 2
A(1,0:2) = 3; // set the line 1 from 0 to 2

cout << " A = " << A << endl;

C = cb*cc'; // outer product
C += 3*cb*cc';
C -= 5i*cb*cc';
cout << " C = " << C << endl;
// this transforms an array into a sparse matrix

matrix B;
B = A;
B=A(I,J); // B(i,j)= A(I(i),J(j))
B=A(I^-1,J^-1); // B(I(i),J(j))= A(i,j)

A = 2.*b*c'; // outer product
cout << " A = " << A << endl;
B = b*c'; // outer product B(i,j) = b(i)*c(j)
B = b*c'; // outer product B(i,j) = b(i)*c(j)
B = (2*b*c')(I,J); // outer product B(i,j) = b(I(i))*c(J(j))
B = (3.*b*c')(I^-1,J^-1); // outer product B(I(i),J(j)) = b(i)*c(j)
cout << "B = (3.*b*c')(I^-1,J^-1) = " << B << endl;

```

the output is

```

b = a(I) : 5
           2         4         4        -3         2

c(I) = a 5
         4        -3         2         1         2

A = 3 4
     1  5  2  1
     3  3  3  1
     4  5  2  4

C = 3 4
      (-50,-40) (-100,-80) (-150,-120) (-200,-160)
      (-100,-80) (-200,-160) (-300,-240) (-400,-320)
      (-150,-120) (-300,-240) (-450,-360) (-600,-480)

A = 3 4
     8  10  12  14
    16  20  24  28
    24  30  36  42

```

4.9.2 Matrix construction and setting

- To change the linear system solver associated to a matrix do

```
set(M,solver=sparse solver);
```

The default solver is GMRES.

- from a variational form: (see section 6.12 page 186 for details)

```

varf vDD(u,v) = int2d(Thm) (u*v*1e-10);
matrix DD=vDD(Lh,Lh);

```

- To set from a constant matrix

```

matrix A =
      [[ 0, 1, 0, 10],
       [ 0, 0, 2, 0],
       [ 0, 0, 0, 3],
       [ 4, 0, 0, 0]];

```

- To set from a block matrix

```

matrix M=[
      [ Asd[0] ,0      ,0      ,0      ,Csd[0] ],
      [ 0      ,Asd[1] ,0      ,0      ,Csd[1] ],
      [ 0      ,0      ,Asd[2] ,0      ,Csd[2] ],
      [ 0      ,0      ,0      ,Asd[3] ,Csd[3] ],
      [ Csd[0]',Csd[1]',Csd[2]',Csd[3]',DD      ]

```

```
];

// to now to pack the right hand side
real[int] bb=[rhssd[0][], rhssd[1][],rhssd[2][],rhssd[3][],rhs1[] ];
set(M,solver=sparse solver);
xx = M^-1 * bb;
[usd[0][],usd[1][],usd[2][],usd[3][],lh[]] = xx; // to dispatch
// the solution on each part.
```

where Asd and Csd are arrays of matrices (from example mortar-DN-4.edp of examples++-tutorial).

- To set or get all the indices and coefficients of the sparse matrix A , let I, J, C be respectively two `int[int]` arrays and a `real[int]` array. The three arrays define the matrix as follows

$$A = \sum_k C[k] M_{I[k],J[k]} \quad \text{where} \quad M_{ab} = (\delta_{ia} \delta_{jb})_{ij}$$

one has: M_{ab} a basic matrix with the only non zero term $m_{ab} = 1$.

One can write `[I,J,C]=A` ; to get all the term of the matrix A (the arrays are automatically resized), and `A=[I,J,C]` ; to change all the term matrices. Note that the size of the matrix is with `n= I.max` and `m=J.max`. Remark that `I, J` is forgotten to build a diagonal matrix, and similarly for the n, m of the matrix.

- matrix renumbering

```
int[int] I(15),J(15); // two array for renumbering
// the aim is to transform a matrix into a sparse matrix

matrix B;
B = A; // copie matrix A
B=A(I,J); // B(i,j) = A(I(i),J(j))
B=A(I^-1,J^-1); // B(I(i),J(j))= A(i,j)
B.resize(10,20); // resize the sparse matrix and remove out of bound
terms
```

where A is a given matrix.

- complex versu real sparse matrix:

```
matrix<complex> C=vv(Xh,Xh);
matrix R=vr(Xh,Xh);
matrix<complex> CR=R; C=R; // create or copy real matrix tp complex
matrix
R=C.im; R=C.re; // get real or imagery part of complex sparse matrix
matrix CI=C.im, CR=C.re; // get real or imagery part of complex sparse
matrix
```

4.9.3 Matrix Operations

The multiplicative operators `*`, `/`, and `%` group left to right.

- `'` is the (unary) right transposition for arrays, the matrix in real cases and Hermitian transpose in complex cases.
- `.*` is the term to term multiply operator.
- `./` is the term to term divide operator.

there are some compound operators also:

- `^-1` is for solving the linear system (example: `b = A^-1 x`)
- `' *` is the compound of transposition and matrix product, so it is the dot product (example `real DotProduct=a'*b`), in complex case you get the Hermitian product, so mathematically we have $a' * b = \bar{a}^T b$.
- `a*b'` is the outer product (example matrix `B=a'*b`)

Example 4.5

```

mesh Th = square(2,1);
fespace Vh(Th,P1);
Vh f,g;
f = x*y;
g = sin(pi*x);
Vh<complex> ff,gg;           // a complex valued finite element function
ff= x*(y+1i);
gg = exp(pi*x*1i);
varf mat(u,v) =
    int2d(Th) (1*dx(u)*dx(v)+2*dx(u)*dy(v)+3*dy(u)*dx(v)+4*dy(u)*dy(v))
    + on(1,2,3,4,u=1);
varf mati(u,v) =
    int2d(Th) (1*dx(u)*dx(v)+2i*dx(u)*dy(v)+3*dy(u)*dx(v)+4*dy(u)*dy(v))
    + on(1,2,3,4,u=1);
matrix A = mat(Vh,Vh); matrix<complex> AA = mati(Vh,Vh);           // a complex
sparse matrix

Vh m0; m0[] = A*f[];
Vh m01; m01[] = A'*f[];
Vh m1; m1[] = f[].*g[];
Vh m2; m2[] = f[]./g[];
cout << "f = " << f[] << endl;
cout << "g = " << g[] << endl;
cout << "A = " << A << endl;
cout << "m0 = " << m0[] << endl;
cout << "m01 = " << m01[] << endl;
cout << "m1 = " << m1[] << endl;
cout << "m2 = " << m2[] << endl;
cout << "dot Product = " << f[]'*g[] << endl;
cout << "hermitien Product = " << ff[]'*gg[] << endl;
cout << "outer Product = " << (A=ff[]*gg[]') << endl;

```

```

cout << "hermitien outer Product = "<< (AA=ff[]*gg[]') << endl;
real[int] diagofA(A.n);
    diagofA = A.diag; // get the diagonal of the matrix
    A.diag = diagofA; // set the diagonal of the matrix
                        // version 2.17 or better ---

int[int] I(1),J(1); real[int] C(1);
[I,J,C]=A; // get of the sparse term of the matrix A (the array are
resized)
cout << " I= " << I << endl;
cout << " J= " << J << endl;
cout << " C= " << C << endl;
A=[I,J,C]; // set a new matrix
matrix D=[diagofA] ; // set a diagonal matrix D from the array diagofA.
cout << " D = " << D << endl;

```

For the second case, it is just a map of the STL¹[26] so no operations on vector are allowed, except the selection of an item .

The transpose or Hermitian conjugation operator is ' as in Matlab or Scilab, so the way to compute the dot product of two array a, b is `real ab= a'*b`.

The resizing of a sparse matrix A is also allowed:

```
A.resize(10,100);
```

Note that the new size can be greater or smaller than the previous size; all new term are set to zero.

On the triangulation of Figure 2.4 this produces the following:

$$A = \begin{bmatrix} 10^{30} & 0.5 & 0. & 30. & -2.5 & 0. \\ 0. & 10^{30} & 0.5 & 0. & 0.5 & -2.5 \\ 0. & 0. & 10^{30} & 0. & 0. & 0.5 \\ 0.5 & 0. & 0. & 10^{30} & 0. & 0. \\ -2.5 & 0.5 & 0. & 0.5 & 10^{30} & 0. \\ 0. & -2.5 & 0. & 0. & 0.5 & 10^{30} \end{bmatrix}$$

$$\begin{aligned}
 \{v\} = f[] &= (0 \ 0 \ 0 \ 0 \ 0.5 \ 1)^T \\
 \{w\} = g[] &= (0 \ 1 \ 1.2 \times 10^{-16} \ 0 \ 1 \ 1.2 \times 10^{-16}) \\
 A * f[] &= (-1.25 \ -2.25 \ 0.5 \ 0 \ 5 \times 10^{29} \ 10^{30})^T \quad (= A\{v\}) \\
 A' * f[] &= (-1.25 \ -2.25 \ 0 \ 0.25 \ 5 \times 10^{29} \ 10^{30})^T \quad (= A^T\{v\}) \\
 f[] * g[] &= (0 \ 0 \ 0 \ 0 \ 0.5 \ 1.2 \times 10^{-16})^T = (v_1 w_1 \ \cdots \ v_M w_M)^T \\
 f[] ./ g[] &= (-NaN \ 0 \ 0 \ -NaN \ 0.5 \ 8.1 \times 10^{15})^T = (v_1/w_1 \ \cdots \ v_M/w_M)^T \\
 f[]' * g[] &= 0.5 \quad (= \{v\}^T \{w\} = \{v\} \cdot \{w\})
 \end{aligned}$$

The output of the I, J, C array:

```

I= 18
      0      0      0      1      1
      1      1      2      2      3
      3      4      4      4      4

```

¹Standard template Library, now part of standard C++

```

      5      5      5
J= 18
      0      1      4      1      2
      4      5      2      5      0
      3      0      1      3      4
      1      4      5
C= 18
1e+30    0.5    -2.5    1e+30    0.5
0.5    -2.5    1e+30    0.5    0.5
1e+30    -2.5    0.5    0.5    1e+30
-2.5    0.5    1e+30

```

The output of a diagonal sparse matrix D (Warning du to fortran interface the indices start on the output at one, but in FreeFem++ in index as in C begin at zero);

```

D = # Sparce Matrix (Morse)
# first line: n m (is symmetric) nbcoef
# after for each nonzero coefficient: i j a_ij where (i,j) \in {1,...,n}x{1,...,m}
6 6 1 6
1      1 1.0000000000000000199e+30
2      2 1.0000000000000000199e+30
3      3 1.0000000000000000199e+30
4      4 1.0000000000000000199e+30
5      5 1.0000000000000000199e+30
6      6 1.0000000000000000199e+30

```

Note 4.3 *The operators $^{-1}$ cannot be used to create a matrix; the following gives an error*

```
matrix AAA = A-1;
```

In examples++-load/lapack.edp a full matrix is inverted using the lapack library and this small dynamic link interface (see for more detail section C page 361).

```

load "lapack"
load "fflapack"
int n=5;
real[int,int] A(n,n),A1(n,n),B(n,n);
for(int i=0;i<n;++i)
for(int j=0;j<n;++j)
  A(i,j)= (i==j) ? n+1 : 1;
cout << A << endl;
A1=A-1; //      def in load "lapack"
cout << A1 << endl;

B=0;
for(int i=0;i<n;++i)
  for(int j=0;j<n;++j)
    for(int k=0;k<n;++k)
      B(i,j) +=A(i,k)*A1(k,j);
cout << B << endl;

//      A1+A-1; attention ne marche pas

inv(A1); //      def in load "fflapack"
cout << A1 << endl;

```

and the output is:


```

5 5
    6    1    1    1    1
    1    6    1    1    1
    1    1    6    1    1
    1    1    1    6    1
    1    1    1    1    6

error:  dgesv_ 0
5 5
0.18 -0.02 -0.02 -0.02 -0.02
-0.02 0.18 -0.02 -0.02 -0.02
-0.02 -0.02 0.18 -0.02 -0.02
-0.02 -0.02 -0.02 0.18 -0.02
-0.02 -0.02 -0.02 -0.02 0.18

5 5
    1 -1.387778781e-17 -1.040834086e-17 3.469446952e-17    0
-1.040834086e-17    1 -1.040834086e-17 -2.081668171e-17    0
3.469446952e-18 -5.551115123e-17    1 -2.081668171e-17 -2.775557562e-17
1.387778781e-17 -4.510281038e-17 -4.857225733e-17    1 -2.775557562e-17
-1.387778781e-17 -9.714451465e-17 -5.551115123e-17 -4.163336342e-17    1

5 5
    6    1    1    1    1
    1    6    1    1    1
    1    1    6    1    1
    1    1    1    6    1
    1    1    1    1    6

```

to compile `lapack.cpp` or `fflapack.cpp` you must have the library `lapack` on you system and try in directory `examples++-load`

```
ff-c++ lapack.cpp -llapack
ff-c++ fflapack.cpp -llapack
```

4.9.4 Other arrays

It is also possible to make an array of FE functions, with the same syntax, and we can treat them as *vector valued function* if we need them. , the syntax for space or vector finite function is

```
int n = 100; // size of the array.
Vh<int> wh(n); // real scalar case
Vh<int> [uh,vh](n); // real vectorial case
Vh<complex>[int] cwh(n); // complex scalar case
Vh<complex>[int] [cuh,cvuh](n); // complex vectorial case
[cuh[2],cvuh[2]] = [x,y]; // set interpolation of index 2.
```

Example 4.6 *In the following example, Poisson's equation is solved for 3 different given functions $f = 1, \sin(\pi x) \cos(\pi y), |x-1||y-1|$, whose solutions are stored in an array of FE function.*

```

mesh Th=square(20,20,[2*x,2*y]);
fespace Vh(Th,P1);
Vh u, v, f;
problem Poisson(u,v) =
    int2d(Th) ( dx(u)*dx(v) + dy(u)*dy(v) )
    + int2d(Th) ( -f*v ) + on(1,2,3,4,u=0) ;
Vh[int] uu(3);                                     // an array of FE function
f=1;                                                  // problem1
Poisson; uu[0] = u;
f=sin(pi*x)*cos(pi*y);                               // problem2
Poisson; uu[1] = u;
f=abs(x-1)*abs(y-1);                                 // problem3
Poisson; uu[2] = u;
for (int i=0; i<3; i++)                             // plots all solutions
    plot(uu[i], wait=true);

```

4.10 Map arrays

For the second case, it is just a map of the STL²[26] so no operations on vector are allowed, except the selection of an item .

```

real[string] map;                                     // a dynamic array
for (i=0;i<10;i=i+1)
{
    tab[i] = i*i;
    cout << i << " " << tab[i] << "\n";
};

map["1"]=2.0;
map[2]=3.0;                                           // 2 is automatically cast to the string "2"

cout << " map[\"1\"] = " << map["1"] << "; " << endl;
cout << " map[2] = " << map[2] << "; " << endl;

```

4.11 Loops

The for and while loops are implemented in FreeFem++ together with break and continue keywords.

In for-loop, there are three parameters; the INITIALIZATION of a control variable, the CONDITION to continue, the CHANGE of the control variable. While CONDITION is true, for-loop continue.

²Standard template Library, now part of standard C++

```
for (INITIALIZATION; CONDITION; CHANGE)
    { BLOCK of calculations }
```

An example below shows a sum from 1 to 10 with result is in `sum`,

```
int sum=0;
for (int i=1; i<=10; i++)
    sum += i;
```

The while-loop

```
while (CONDITION) {
    BLOCK of calculations or change of control variables
}
```

is executed repeatedly until `CONDITION` become false. The sum from 1 to 10 can also be computed by `while` as follows,

```
int i=1, sum=0;
while (i<=10) {
    sum += i; i++;
}
```

We can exit from a loop in midstream by `break`. The `continue` statement will pass the part from `continue` to the end of the loop.

Example 4.7

```
for (int i=0; i<10; i=i+1)
    cout << i << "\n";
real eps=1;
while (eps>1e-5)
{
    eps = eps/2;
    if( i++ <100) break;
    cout << eps << endl;}

for (int j=0; j<20; j++) {
    if (j<10) continue;
    cout << "j = " << j << endl;
}
```

4.12 Input/Output

The syntax of input/output statements is similar to C++ syntax. It uses `cout`, `cin`, `endl`, `<<`, `>>`.

To write to (resp. read from) a file, declare a new variable `ofstream ofile("filename");` or `ofstream ofile("filename", append);` (resp. `ifstream ifile("filename");`) and use `ofile` (resp. `ifile`) as `cout` (resp. `cin`).

The word `append` in `ofstream ofile("filename", append|binary);` means opening a file in `append` and `binary` mode.

Note 4.4 *The file is closed at the exit of the enclosing block,*

Example 4.8

```

int i;
cout << " std-out" << endl;
cout << " enter i= ? ";
cin >> i ;
{
    ofstream f("toto.txt");
    f << i << "coucou'\n";
}; // close the file f because the variable f is delete

{
    ifstream f("toto.txt");
    f >> i;
}
{
    ofstream f("toto.txt",append); // to append to the existing file "toto.txt"
    f << i << "coucou'\n";
}; // close the file f because the variable f is delete

cout << i << endl;

```

Some functions are available to format the output.

- `int nold=f.precision(n)` Sets the number of digits printed to the right of the decimal point. This applies to all subsequent floating point numbers written to that output stream. However, this won't make floating-point "integers" print with a decimal point. It's necessary to use `fixed` for that effect.
- `f.scientific` Formats floating-point numbers in scientific notation (`d.dddEdd`)
- `f.fixed` Used fixed point notation (`d.ddd`) for floating-point numbers. Opposite of `scientific`.
- `f.showbase` Converts insertions to an external form that can be read according to the C++ lexical conventions for integral constants. By default, `showbase` is not set.
- `f.noshowbase` unset `showbase` flags
- `f.showpos` inserts a plus sign (+) into a decimal conversion of a positive integral value.
- `f.noshowpos` unset `showpos` flags
- `f.default` reset all the previous flags (`fmtflags`) to the default expect precision.

Where `f` is output stream descriptor, for example `cout`.

Remark, all these methods except the first return the stream `f`, so they can be chained as in

```
cout.scientific.showpos << 3 << endl;
```

4.12.1 Script arguments

There is a very useful predefined array in FreeFem++ `ARGV` that contains all the arguments of the script used in the command line. The following code prints out the first three of these arguments:

```

// version 3.8-1
for(int i=0;i<ARGV.n;++i)
{
    cout << ARGV[i] << endl;
}

```

And to get argument unused in `getARGV.idp` include script file,

```

getARGV(n,defaultvalue)    // get the nth parameter unused if exist (n = 1,
...)
getARGV(after,defaultvalue) // get the arg after the string after if exist

```

The type of default value can be int, real, string,

4.13 preprocessor

The preprocessor handles directives for source file inclusion (`include "script-name.idp"`), macro definitions.

There are two types of macros, object-like and function-like. Object-like macros do not take parameters; function-like macros do. The generic syntax for declaring an identifier as a macro of each type is, respectively,

```

macro <identifier>() <replacement token list>    // EOM a // comment to end
the macro
macro <identifier>(<parameter list>) <replacement token list>    // EOM

```

An example of macro without parameter

```

macro xxx() {real i=0;int j=0;cout << i << " " << j << endl;}    //
xxx /* replace xxx by the <replacement token list> */

```

The FreeFem++ code associated:

```

1 : // macro without parameter
2 : macro xxx {real i=0;int j=0;cout << i << " " << j << endl;}//
3 :
4 :             {real i=0;int j=0;cout << i << " " << j << endl;}

```

An example of macro parameter

```

macro toto(i) i    //
// quoting parameter the {} are remove
toto({real i=0;int j=0;cout << i << " " << j << endl;})
// and only one level of {} are remove
toto({{real i=0;int j=0;cout << i << " " << j << endl;}})

```

The FreeFem++ code created :

```

6 : macro toto(i )    i //
8 : // quoting parameter the \{\} are remove
9 :         real i=0;int j=0;cout << i << " " << j << endl;
10 : // and only one level of \{\} are remove
11 :         {real i=0;int j=0;cout << i << " " << j << endl;}

```

Use a macro as parameter of macro to transforme full matrix in formal array like in :

```

real[int,int] CC(7,7),EE(6,3),EEps(4,4);

macro VIL6(v,i) [ v(1,i), v(2,i),v(4,i), v(5,i),v(6,i) ]           // EOM
macro VIL3(v,i) [ v(1,i), v(2,i) ]                                   // EOM
                                                                    // apply v on array element :
macro VV6(v,vv) [ v(vv,1), v(vv,2),
v(vv,4), v(vv,5), v(vv,6) ]                                         // EOM
macro VV3(v,vv) [ v(vv,1), v(vv,2) ]                                 // EOM
                                                                    // so formal matrix to build problem..

func C5x5 = VV6(VIL6,CC);
func E5x2 = VV6(VIL3,EE);
func Eps = VV3(VIL3,EEps);

```

The FreeFem++ code created :

```

16 : real[int,int] CC(7,7),EE(6,3),EEps(4,4);
17 :
18 : macro VIL6(v,i) [ v(1,i), v(2,i),v(4,i), v(5,i),v(6,i) ] //
19 : macro VIL3(v,i) [ v(1,i), v(2,i) ] // EOM
20 : // apply v on array element :
21 : macro VV6(v,vv) [ v(vv,1), v(vv,2),
22 : v(vv,4), v(vv,5), v(vv,6) ] // EOM
23 : macro VV3(v,vv) [ v(vv,1), v(vv,2) ] // EOM
24 : // so formal matrix to build problem..
25 : func C5x5 =
1 : [ [ CC(1,1), CC(2,1),CC(4,1), CC(5,1),CC(6,1) ]
1 : [ CC(1,2), CC(2,2),CC(4,2), CC(5,2),CC(6,2) ] ,
[ CC(1,4), CC(2,4),CC(4,4), CC(5,4),CC(6,4) ] ,
[ CC(1,5), CC(2,5),CC(4,5), CC(5,5),CC(6,5) ] ,
[ CC(1,6), CC(2,6),CC(4,6), CC(5,6),CC(6,6) ]
26 : func E5x2 =
1 : [ [ EE(1,1), EE(2,1) ] , [ EE(1,2), EE(2,2) ]
1 : [ EE(1,4), EE(2,4) ] , [ EE(1,5), EE(2,5) ] ,
[ EE(1,6), EE(2,6) ] ] ;
27 : func Eps = [ [ EEps(1,1), EEps(2,1) ] ,
[ EEps(1,2), EEps(2,2) ] ] ;
28 :

```

finally the operator # to do concatenation of parameter: to build vectorial operation, like in

```

macro div(u) (dx(u#1)+ dy(u#2)) // EOM
mesh Th=square(2,2); fespace Vh(Th,P1);

```

```
Vh v1=x,v2=y;
cout << int2d(Th) (div(v)) << endl;
```

The FreeFem++ code created :

```
31 : macro div(u )      (dx(u#1)+ dy(u#2)) //EOM
32 : mesh Th=square(2,2); fespace Vh(Th,P1);
33 : Vh v1=x,v2=y;
34 : cout << int2d(Th) (      (dx(v1)+ dy(v2)) ) << endl;
```

And to finish a amazing test to verified the quoting :

```
macro foo(i,j,k) i j k                                // EOM
foo(,,)                                                  // empty line
foo( {int [] , {int] a(10),{}});)
```

the result:

```
36 : macro foo(i,j,k )      i j k//EOM
37 :           // empty line
38 :           int [ int] a(10 );
```

To defined macro in a macro you can use the two new word NewMacro , EndMacro key word to set and and claose de macro definition (version 3.11, and not well tested).

4.14 Exception handling

In the version 2.3 of FreeFem++ , exception handing was added as in C++. But today only the C++ exceptions are caught. Note that in C++ all the errors attached to ExecError, assert, exit, ... call exceptions too so it may be hard to find the cause of the error. The exceptions handle all ExecError:

Example 4.9 *A simple example: catch a division by zero:*

```
real a;
try {
  a=1./0.;
}
catch (...) // in versions > 2.3 all exceptions can be caught
{
  cout << " Catch an ExecError " << endl;
  a =0;
}
```

The output is

```
1/0 : d d d
      current line = 3
Exec error : Div by 0
      -- number :1
Try:: catch (...) exception
Catch an ExecError
```

Example 4.10 : *a more realistic example with a none invertible matrix:*

```

int nn=5          ;
mesh Th=square(nn,nn);
verbosity=5;
fespace Vh(Th,P1);                                // P1 FE space
Vh uh,vh;                                           // unkown and test function.
func f=1;                                           // right hand side function
func g=0;                                           // boundary condition function
real cpu=clock();
problem laplace(uh,vh,solver=Cholesky,tolpivot=1e-6) = //
definition of the problem
    int2d(Th) ( dx(uh)*dx(vh) + dy(uh)*dy(vh) ) // bilinear form
    + int2d(Th) ( -f*vh ) // linear form
;

try {
    cout << " Try Cholesky \n";
    laplace; // solve the problem
    plot(uh); // to see the result
    cout << "-- lap Cholesky " << nn << "x" << nn << " : " << -cpu+clock()
        << " s, max =" << uh[].max << endl;
}
catch(...) { // catch all
    cout << " Catch cholesky PB " << endl;
}

```

The output is

```

-- square mesh : nb vertices =36 , nb triangles = 50 ...
Nb of edges on Mortars = 0
Nb of edges on Boundary = 20, neb = 20
Nb Mortars 0
number of real boundary edges 20
Number of Edges = 85
Number of Boundary Edges = 20 neb = 20
Number of Mortars Edges = 0
Nb Of Mortars with Paper Def = 0 Nb Of Mortars = 0 ...
Nb Of Nodes = 36
Nb of DF = 36
Try Cholesky
-- Change of Mesh 0 0x312e9e8
Problem(): initmat 1 VF (discontinuous Galerkin) = 0
-- SizeOfSkyline =210
-- size of Matrix 196 Bytes skyline =1
-- discontinous Galerkin =0 size of Mat =196 Bytes
-- int in Optimized = 1, ...
all
-- boundary int Optimized = 1, all
ERREUR choleskypivot (35)= -1.23124e-13 < 1e-06
current line = 28
Exec error : FATAL ERREUR dans ../femlib/MatriceCreuse_tpl.hpp

```



```
cholesky line:
  -- number :545
  catch an erreur in solve => set sol = 0 !!!!!!!
Try:: catch (...) exception
  Catch cholesky PB
```


Chapter 5

Mesh Generation

5.1 Commands for Mesh Generation

Let us begin with the two important keywords **border** and **buildmesh**

All examples in this section come from the files `mesh.edp` and `tablefunction.edp`.

5.1.1 Square

The command “**square**” triangulates the unit square. The following

```
mesh Th = square(4, 5);
```

generates a 4×5 grid in the unit square $[0, 1]^2$. The labels of the boundaries are shown in Fig. 5.1. To construct a $n \times m$ grid in the rectangle $[x_0, x_1] \times [y_0, y_1]$, proceeds as follows:

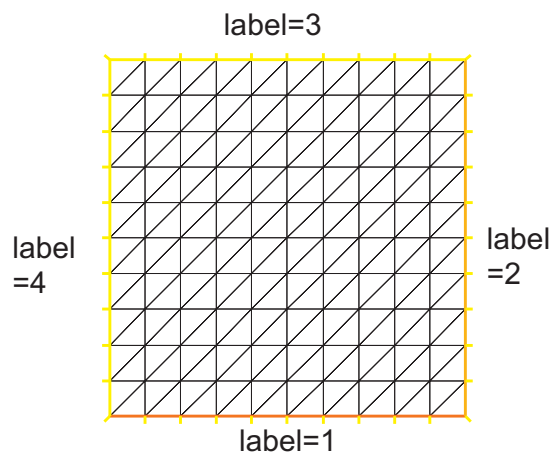


Figure 5.1: Boundary labels of the mesh by `square(10, 10)`

write

```
real x0=1.2, x1=1.8;  
real y0=0, y1=1;  
int n=5, m=20;  
mesh Th=square(n, m, [x0+(x1-x0)*x, y0+(y1-y0)*y]);
```

Note 5.1 Adding the named parameter $flags=icase$ with $icase$:

0 will produce a mesh where all quads are split with diagonal $x - y = cte$

1 will produce Union Jack flag type of mesh.

2 will produce a mesh where all quads are split with diagonal $x + y = cte$ (v 3.8)

3 same as case 0 except in two corners such that no triangle with 3 vertices on boundary (v 3.8)

4 same as case 2 except in two corners such that no triangle with 3 vertices on boundary (v 3.8)

```
mesh Th=square(n,m,[x0+(x1-x0)*x,y0+(y1-y0)*y],flags=icase);
```

Adding the named parameter $label=labs$ will change the 4 default label numbers to $labs[i-1]$, for example `int[int] labs=[11,12,13,14]`, and adding the named parameter $region=10$ will change the region number to 10, for instance (v 3.8).

To see all these fags at work, try the file `examples++/square-mesh.edp` :

```
for (int i=0;i<5;++i)
{
    int[int] labs=[11,12,13,14];
    mesh Th=square(3,3,flags=i,label=labs,region=10);
    plot(Th,wait=1,cmm=" square flags = "+i );
}
```

5.1.2 Border

Boundaries are defined piecewise by parametrized curves. The pieces can only intersect at their endpoints, but it is possible to join more than two endpoints. This can be used to structure the mesh if an area thouches a border and create new regions by dividing larger ones:

```
int upper = 1;
int others = 2;
int inner = 3;

border C01(t=0,1){x = 0;          y = -1+t;          label = upper;}
border C02(t=0,1){x = 1.5-1.5*t; y = -1;            label = upper;}
border C03(t=0,1){x = 1.5;        y = -t;            label = upper;}
border C04(t=0,1){x = 1+0.5*t;    y = 0;            label = others;}
border C05(t=0,1){x = 0.5+0.5*t; y = 0;            label = others;}
border C06(t=0,1){x = 0.5*t;      y = 0;            label = others;}
border C11(t=0,1){x = 0.5;        y = -0.5*t;        label = inner;}
border C12(t=0,1){x = 0.5+0.5*t; y = -0.5;        label = inner;}
```

```

border C13(t=0,1){x = 1;          y = -0.5+0.5*t;  label = inner;}

int n = 10;
plot (C01(-n)+C02(-n)+C03(-n)+C04(-n)+C05(-n)+C06(-n)+
      C11(n)+C12(n)+C13(n), wait=true);

mesh Th = buildmesh(C01(-n)+C02(-n)+C03(-n)+C04(-n)+C05(-n)+C06(-n)+
                   C11(n)+C12(n)+C13(n));

plot(Th, wait=true); // figure 5.3

cout << "Part 1 has region number " << Th(0.75, -0.25).region << endl;
cout << "Part 2 has region number " << Th(0.25, -0.25).region << endl;

```

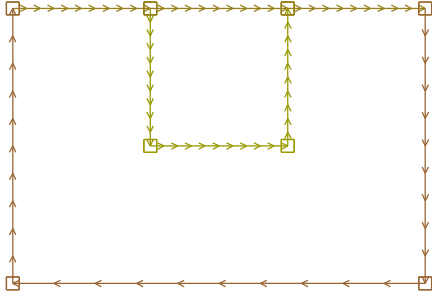


Figure 5.2: Multiple border ends intersect

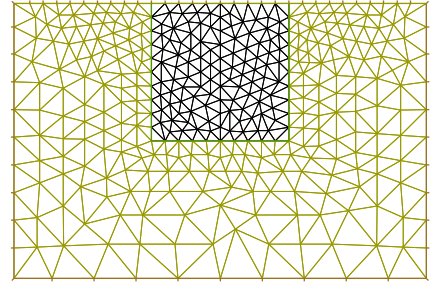


Figure 5.3: Generated mesh

Triangulation keywords assume that the domain is defined as being on the *left* (resp *right*) of its oriented parameterized boundary

$$\Gamma_j = \{(x, y) \mid x = \varphi_x(t), y = \varphi_y(t), a_j \leq t \leq b_j\}$$

To check the orientation plot $t \mapsto (\varphi_x(t), \varphi_y(t))$, $t_0 \leq t \leq t_1$. If it is as in Fig. 5.4, then the domain lies on the shaded area, otherwise it lies on the opposite side

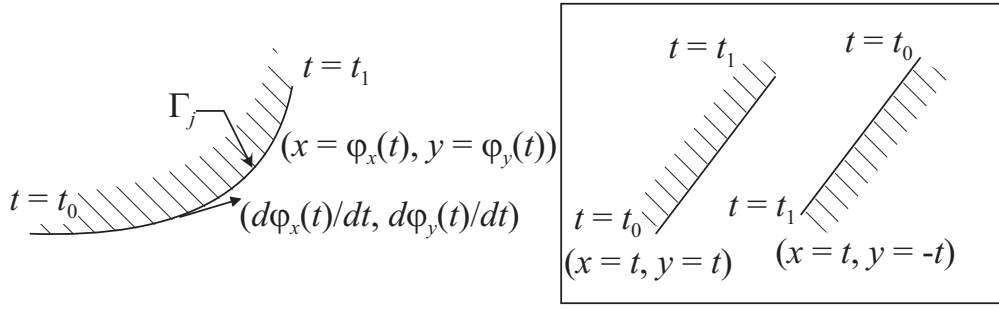
The general expression to define a triangulation with `buildmesh` is

```
mesh Mesh_Name = buildmesh(Γ1(m1) + ⋯ + ΓJ(mJ) OptionalParameter);
```

where m_j are positive or negative numbers to indicate how many vertices should be on Γ_j , $\Gamma = \cup_{j=1}^J \Gamma_j$, and the optional parameter (separated with comma) can be

nbvx=<int value> , to set the maximal number of vertices in the mesh.

fixeborder=<bool value> , to say if the mesh generator can change the boundary mesh or not (by default the boundary mesh can change; beware that with periodic boundary conditions (see. 6), it can be dangerous .

Figure 5.4: Orientation of the boundary defined by $(\phi_x(t), \phi_y(t))$

The orientation of boundaries can be changed by changing the sign of m_j . The following example shows how to change the orientation. The example generates the unit disk with a small circular hole, and assign “1” to the unit disk (“2” to the circle inside). The boundary label must be non-zero, but it can also be omitted.

```

1: border a(t=0,2*pi){ x=cos(t); y=sin(t);label=1;}
2: border b(t=0,2*pi){ x=0.3+0.3*cos(t); y=0.3*sin(t);label=2;}
3: plot (a(50)+b(+30)) ; // to see a plot of the border mesh
4: mesh Thwithouthole= buildmesh (a(50)+b(+30));
5: mesh Thwithhole = buildmesh (a(50)+b(-30));
6: plot (Thwithouthole,wait=1,ps="Thwithouthole.eps"); // figure 5.5
7: plot (Thwithhole,wait=1,ps="Thwithhole.eps"); // figure 5.6

```

Note 5.2 Notice that the orientation is changed by “b(-30)” in 5th line. In 7th line, $ps="fileName"$ is used to generate a postscript file with identification shown on the figure.

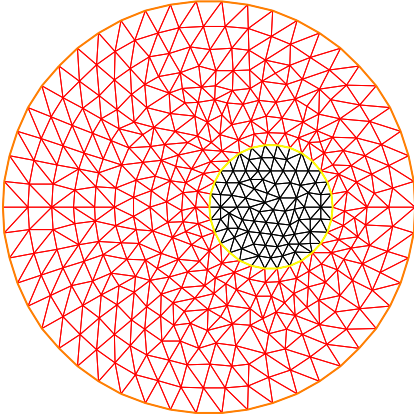


Figure 5.5: mesh without hole

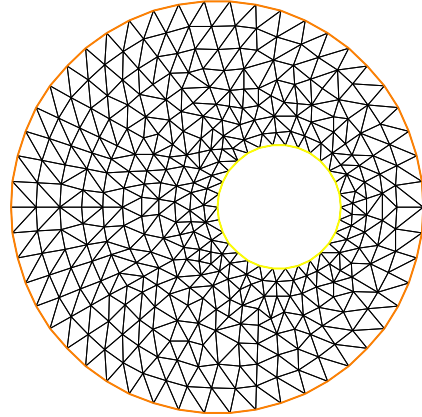


Figure 5.6: mesh with hole

Note 5.3 Borders are evaluated only at the time `plot` or `buildmesh` is called so the global variable are defined at this time and here since r is changed between the two border calls the following code will not work because the first border will be computed with $r=0.3$:

```

real r=1;    border a(t=0,2*pi){ x=r*cos(t); y=r*sin(t);label=1;}
r=0.3      ;    border b(t=0,2*pi){ x=r*cos(t); y=r*sin(t);label=1;}
mesh Thwithhole = buildmesh(a(50)+b(-30));      //    bug (a trap) because
                                                    //    the two circle have the same radius = 0.3

```

5.1.3 Multy Border

(Version 3.30)

Some time it can be useful to to make array of border, but unfortunately it is incompatible with the FreeFem++ syntax. So to bypass this , thes dea is small, if the number of segment of the discretization n is a array, we make a loop on all the value of the array, and the index variable i of the loop is defined after parameter definition, like in `border a(t=0,2*pi;i)`

A first very small example:

```

1: border a(t=0,2*pi;i){ x=(i+1)*cos(t); y=(i+1)*sin(t);label=1;}
2: int[int] nn=[10,20,30];
3: plot (a(nn));      //    plot 3 circle with 10,20,30 points ..

```

And more complex exemple (take from `mesh.edp` example) to defined a square with small circies:

```

                                                    //    multy border syntax version 3.30 avril 2014 ...
real[int] xx=[0,1,1,0],
yy=[0,0,1,1];

                                                    //    radius, centre of the 4 circles ..
real[int] RC=[ 0.1, 0.05, 0.05, 0.1],
XC= [0.2,0.8,0.2,0.8],
YC= [0.2,0.8,0.8,0.2];
int[int] NC=[-10,-11,-12,13];      //    list number of ± segments
                                     //    of the 4 circles borders

border bb(t=0,1;i)
{
    //    i is the the index variable of the multi border loop
    int ii = (i+1)%4; real t1 = 1-t;
    x = xx[i]*t1 + xx[ii]*t;
    y = yy[i]*t1 + yy[ii]*t;
    label = 0; ;
}

border cc(t=0,2*pi;i)
{
    x = RC[i]*cos(t)+XC[i];
    y = RC[i]*sin(t)+YC[i];
    label = i+1;
}
int[int] nn=[4,4,5,7]; //    4 border , with 4,4,5,7 segment respectively .
plot (bb(nn),cc(NC),wait=1);
mesh th= buildmesh(bb(nn)+cc(NC)) ;
plot (th,wait=1);

```

5.1.4 Data Structures and Read/Write Statements for a Mesh

Users who want to read a triangulation made elsewhere should see the structure of the file generated below:

```
border C(t=0,2*pi) { x=cos(t); y=sin(t); }
mesh Th = buildmesh(C(10));
savemesh("mesh_sample.msh");
```

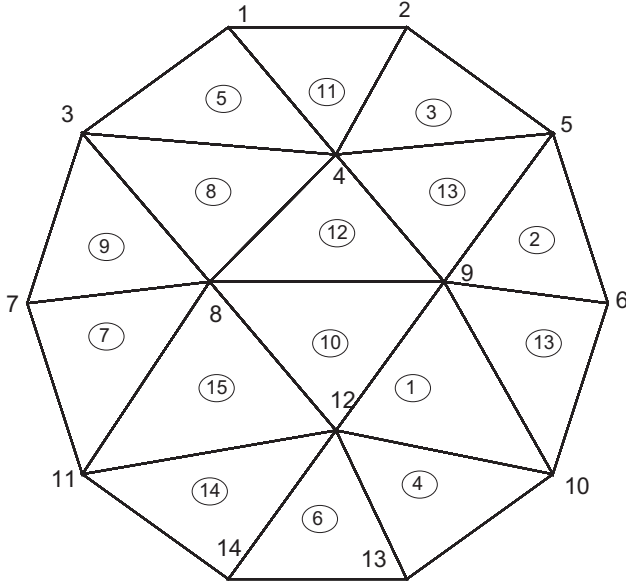
the mesh is shown on Fig. 5.7.

The informations about Th are saved in the file “mesh_sample.msh”. whose structure is shown on Table 5.1.

There n_v denotes the number of vertices, n_t number of triangles and n_s the number of edges on boundary.

For each vertex q^i , $i = 1, \dots, n_v$, denote by (q_x^i, q_y^i) the x -coordinate and y -coordinate.

Each triangle T_k , $k = 1, \dots, 10$ has three vertices $q^{k_1}, q^{k_2}, q^{k_3}$ that are oriented counterclockwise. The boundary consists of 10 lines L_i , $i = 1, \dots, 10$ whose end points are q^{i_1}, q^{i_2} .



In the left figure, we have the following.

$$n_v = 14, n_t = 16, n_s = 10$$

$$q^1 = (-0.309016994375, 0.951056516295)$$

$$\vdots \quad \vdots \quad \vdots$$

$$q^{14} = (-0.309016994375, -0.951056516295)$$

The vertices of T_1 are q^9, q^{12}, q^{10} .

$$\vdots \quad \vdots \quad \vdots$$

The vertices of T_{16} are q^9, q^{10}, q^6 .

The edge of 1st side L_1 are q^6, q^5 .

$$\vdots \quad \vdots \quad \vdots$$

The edge of 10th side L_{10} are q^{10}, q^6 .

Figure 5.7: mesh by `buildmesh(C(10))`

In FreeFem++ there are many mesh file formats available for communication with other tools such as emc2, modulef.. (see Section 12), The extension of a file implies its format. More details can be found on the file format .msh in the article by F. Hecht ”bamg : a bidimensional anisotropic mesh generator” (downloadable from the FreeFem web site.)

A mesh file can be read into FreeFem++ except that the names of the borders are lost and only their reference numbers are kept. So these borders have to be referenced by the number which corresponds to their order of appearance in the program, unless this number is overwritten by the keyword ”label”. Here are some examples:

```
border floor(t=0,1){ x=t; y=0; label=1;}; // the unit square
border right(t=0,1){ x=1; y=t; label=5;};
```


Content of the file	Explanation
14 16 10 -0.309016994375 0.951056516295 1 0.309016994375 0.951056516295 1 : -0.309016994375 -0.951056516295 1	n_v n_t n_e q_x^1 q_y^1 boundary label=1 q_x^2 q_y^2 boundary label=1 q_x^{14} q_y^{14} boundary label=1
9 12 10 0 5 9 6 0 ... 9 10 6 0	1_1 1_2 1_3 region label=0 2_1 2_2 2_3 region label=0 16_1 16_2 16_3 region label=0
6 5 1 5 2 1 ... 10 6 1	1_1 1_2 boundary label=1 2_1 2_2 boundary label=1 10_1 10_2 boundary label=1

Table 5.1: The structure of “mesh_sample.msh”

```

border ceiling(t=1,0){ x=t; y=1; label=5;};
border left(t=1,0){ x=0; y=t; label=5;};
int n=10;
mesh th= buildmesh(floor(n)+right(n)+ceiling(n)+left(n));
savemesh(th,"toto.am_fmt"); // "formatted Marrocco" format
savemesh(th,"toto.Th"); // "bamg"-type mesh
savemesh(th,"toto.msh"); // freefem format
savemesh(th,"toto.nopo"); // modulef format see [10]
mesh th2 = readmesh("toto.msh"); // read the mesh

```

Example 5.1 (Readmesh.edp) **border** floor(t=0,1){ x=t; y=0; label=1;}; // the unit square

```

border right(t=0,1){ x=1; y=t; label=5;};
border ceiling(t=1,0){ x=t; y=1; label=5;};
border left(t=1,0){ x=0; y=t; label=5;};
int n=10;
mesh th= buildmesh(floor(n)+right(n)+ceiling(n)+left(n));
savemesh(th,"toto.am_fmt"); // format "formatted Marrocco"
savemesh(th,"toto.Th"); // format database db mesh "bamg"
savemesh(th,"toto.msh"); // format freefem
savemesh(th,"toto.nopo"); // modulef format see [10]
mesh th2 = readmesh("toto.msh");
fespace femp1(th,P1);
femp1 f = sin(x)*cos(y),g;
{ // save solution
ofstream file("f.txt");
file << f[] << endl;
} // close the file (end block)
{ // read
ifstream file("f.txt");
file >> g[] ;
} // close reading file (end block)

```

```

fespace Vh2(th2,P1);
Vh2 u,v;
plot(g);
// find u such that
//  $u + \Delta u = g$  in  $\Omega$  ,
//  $u = 0$  on  $\Gamma_1$  and  $\frac{\partial u}{\partial n} = g$  on  $\Gamma_2$ 
solve pb(u,v) =
    int2d(th) ( u*v - dx(u)*dx(v)-dy(u)*dy(v) )
    + int2d(th) (-g*v)
    + int1d(th,5) ( g*v) //  $\frac{\partial u}{\partial n} = g$  on  $\Gamma_2$ 
    + on(1,u=0) ;
plot (th2,u);

```

5.1.5 Mesh Connectivity

The following example explains methods to obtain mesh information.

```

{ // get mesh information (version 1.37)
mesh Th=square(2,2); // get data of the mesh

int nbtriangles=Th.nt;
cout << " nb of Triangles = " << nbtriangles << endl;
for (int i=0;i<nbtriangles;i++)
    for (int j=0; j <3; j++)
        cout << i << " " << j << " Th[i][j] = "
            << Th[i][j] << " x = " << Th[i][j].x << " , y= " << Th[i][j].y
            << " , label=" << Th[i][j].label << endl;

// Th(i) return the vertex i of Th
// Th[k] return the triangle k of Th

fespace femp1(Th,P1);
femp1 Thx=x,Thy=y; // hack of get vertex coordinates
// get vertices information :

int nbvertices=Th.nv;
cout << " nb of vertices = " << nbvertices << endl;
for (int i=0;i<nbvertices;i++)
    cout << "Th(" <<i << ") : " // << endl;
    << Th(i).x << " " << Th(i).y << " " << Th(i).label // v 2.19
    << " old method: " << Thx[i] << " " << Thy[i] << endl;

// method to find information of point (0.55,0.6)

int it00 = Th(0.55,0.6).nuTriangle; // then triangle number
int nr00 = Th(0.55,0.6).region; //

// info of a triangle
real area00 = Th[it00].area; // new in version 2.19
real nrr00 = Th[it00].region; // new in version 2.19
real nll00 = Th[it00].label; // same as region in this case.

// Hack to get a triangle containing point x,y
// or region number (old method)

```

```

// -----
fespace femp0(Th,P0);
femp0 nuT; // a P0 function to get triangle numbering
for (int i=0;i<Th.nt;i++)
    nuT[i]=i;
femp0 nuReg=region; // a P0 function to get the region number
// inquire
int it0=nuT(0.55,0.6); // number of triangle Th's containing (0.55,0,6);
int nr0=nuReg(0.55,0.6); // number of region of Th's containing (0.55,0,6);

// dump
// -----

cout << " point (0.55,0,6) :triangle number " << it00 << " " << it00
    << ", region = " << nr0 << " == " << nr00 << ", area K " << area00 << endl;

// new method to get boundary information and mesh adjacent

int k=0,l=1,e=1;
Th.nbe ; // return the number of boundary element
Th.be(k); // return the boundary element  $k \in \{0,...,Th.nbe-1\}$ 
Th.be(k)[1]; // return the vertices  $l \in \{0,1\}$  of boundary elmt k
Th.be(k).Element ; // return the triangle containing the boundary elmt
k
Th.be(k).whoinElement ; // return the edge number of triangle
containing

// the boundary elmt k
Th[k].adj(e) ; // return adjacent triangle to k by edge e, and change
// the value of e to the corresponding edge in the adjacent triangle
Th[k] == Th[k].adj(e) // non adjacent triangle return the same
Th[k] != Th[k].adj(e) // true adjacent triangle

cout << " print mesh connectivity " << endl;
int nbelement = Th.nt;
for (int k=0;k<nbelement;++k)
    cout << k << " : " << int(Th[k][0]) << " " << int(Th[k][1])
        << " " << int(Th[k][2])
        << " , label " << Th[k].label << endl;
//

for (int k=0;k<nbelement;++k)
    for (int e=0,ee;e<3;++e)
        // remark FH hack: set ee to e, and ee is change by method adj,
        // in () to make difference with named parameters.
        cout << k << " " << e << " <=> " << int(Th[k].adj((ee=e))) << " " << ee
            << " adj: " << ( Th[k].adj((ee=e)) != Th[k]) << endl;
        // note : if k == int(Th[k].adj(ee=e)) not adjacent element

int nbboundaryelement = Th.nbe;

for (int k=0;k<nbboundaryelement;++k)
    cout << k << " : " << Th.be(k)[0] << " " << Th.be(k)[1] << " , label "
        << Th.be(k).label << " tria " << int(Th.be(k).Element)
        << " " << Th.be(k).whoinElement << endl;

```

```

}
```

the output is:

```

-- square mesh : nb vertices =9 , nb triangles = 8 , nb boundary edges 8
Nb of Vertices 9 , Nb of Triangles 8
Nb of edge on user boundary 8 , Nb of edges on true boundary 8
number of real boundary edges 8
nb of Triangles = 8
0 0 Th[i][j] = 0 x = 0 , y= 0, label=4
0 1 Th[i][j] = 1 x = 0.5 , y= 0, label=1
0 2 Th[i][j] = 4 x = 0.5 , y= 0.5, label=0
...
6 0 Th[i][j] = 4 x = 0.5 , y= 0.5, label=0
6 1 Th[i][j] = 5 x = 1 , y= 0.5, label=2
6 2 Th[i][j] = 8 x = 1 , y= 1, label=3
7 0 Th[i][j] = 4 x = 0.5 , y= 0.5, label=0
7 1 Th[i][j] = 8 x = 1 , y= 1, label=3
7 2 Th[i][j] = 7 x = 0.5 , y= 1, label=3
Nb Of Nodes = 9
Nb of DF = 9
-- vector function's bound 0 1
-- vector function's bound 0 1
nb of vertices = 9
Th(0) : 0 0 4 old method: 0 0
Th(1) : 0.5 0 1 old method: 0.5 0
...
Th(7) : 0.5 1 3 old method: 0.5 1
Th(8) : 1 1 3 old method: 1 1
Nb Of Nodes = 8
Nb of DF = 8

print mesh connectivity
0 : 0 1 4 , label 0
1 : 0 4 3 , label 0
...
6 : 4 5 8 , label 0
7 : 4 8 7 , label 0
0 0 <=> 3 1 adj: 1
0 1 <=> 1 2 adj: 1
0 2 <=> 0 2 adj: 0
...
6 2 <=> 3 0 adj: 1
7 0 <=> 7 0 adj: 0
7 1 <=> 4 0 adj: 1
7 2 <=> 6 1 adj: 1
0 : 0 1 , label 1 tria 0 2
1 : 1 2 , label 1 tria 2 2
...
6 : 0 3 , label 4 tria 1 1
7 : 3 6 , label 4 tria 5 1
```

5.1.6 The keyword "triangulate"

FreeFem++ is able to build a triangulation from a set of points. This triangulation is a Delaunay mesh of the convex hull of the set of points. It can be useful to build a mesh from a table function.

The coordinates of the points and the value of the table function are defined separately with rows of the form: $x \ y \ f(x,y)$ in a file such as:

```
0.51387 0.175741 0.636237
0.308652 0.534534 0.746765
0.947628 0.171736 0.899823
0.702231 0.226431 0.800819
0.494773 0.12472 0.580623
0.0838988 0.389647 0.456045
.....
```

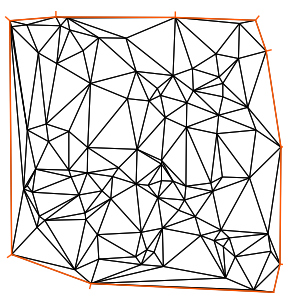


Figure 5.8: Delaunay mesh of the convex hull of point set in file xyf

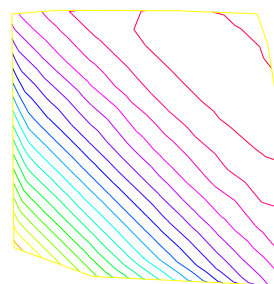


Figure 5.9: Isovalue of table function

The third column of each line is left untouched by the `triangulate` command. But you can use this third value to define a table function with rows of the form: $x \ y \ f(x,y)$. The following example shows how to make a mesh from the file "xyf" with the format stated just above. The command `triangulate` command use only use 1st and 2nd rows.

```
mesh Thxy=triangulate("xyf"); // build the Delaunay mesh of the convex hull
// points are defined by the first 2 columns of file xyf
plot(Thxy,ps="Thxyf.ps"); // (see figure 5.8)

fespace Vhxy(Thxy,P1); // create a P1 interpolation
Vhxy fxy; // the function

// reading the 3rd row to define the function
{ ifstream file("xyf");
  real xx,yy;
  for(int i=0;i<fxy.n;i++)
    file >> xx >>yy >> fxy[i];
}
plot(fxy,ps="xyf.eps"); // plot the function (see figure 5.9)
```

One new way to build a mesh is to have two arrays one the x values and the other for the y values (version 2.23-2):

```
Vhxy xx=x,yy=y; // to set two arrays for the x's and y's
mesh Th=triangulate(xx[],yy[]);
```

5.2 Boundary FEM Spaces Built as Empty Meshes

To define a Finite Element space on a boundary, we came up with the idea of a mesh with no internal points (call empty mesh). It can be useful to handle Lagrange multipliers in mixed and mortar methods.

So the function `emptymesh` remove all the internal points of a mesh except points on internal boundaries.

```
{ // new stuff 2004 emptymesh (version 1.40)
  // -- useful to build Multiplicator space
  // build a mesh without internal point
  // with the same boundary
  // -----

  assert(version>=1.40);
  border a(t=0,2*pi){ x=cos(t); y=sin(t);label=1;}
  mesh Th=buildmesh(a(20));
  Th=emptymesh(Th);
  plot(Th,wait=1,ps="emptymesh-1.eps"); // see figure 5.10
}
```

It is also possible to build an empty mesh of a pseudo subregion with `emptymesh(Th,ssd)` using the set of edges of the mesh `Th`; a edge e is in this set if with the two adjacent triangles $e = t1 \cap t2$ and $ssd[T1] \neq ssd[T2]$ where `ssd` refers to the pseudo region numbering of triangles, when they are stored in an `int[int]` array of size the number of triangles.

```
{ // new stuff 2004 emptymesh (version 1.40)
  // -- useful to build Multiplicator space
  // build a mesh without internal point
  // of pseudo sub domain
  // -----

  assert(version>=1.40);
  mesh Th=square(10,10);
  int[int] ssd(Th.nt);
  for(int i=0;i<ssd.n;i++) // build the pseudo region numbering
  { // because 2 triangle per quad
    int iq=i/2; //
    int ix=iq%10; //
    int iy=iq/10; //
    ssd[i]= 1 + (ix>=5) + (iy>=5)*2;
  }
  Th=emptymesh(Th,ssd); // build empty with
  // all edge  $e = T1 \cap T2$  and  $ssd[T1] \neq ssd[T2]$ 
  plot(Th,wait=1,ps="emptymesh-2.eps"); // see figure 5.11
  savemesh(Th,"emptymesh-2.msh");
}
```

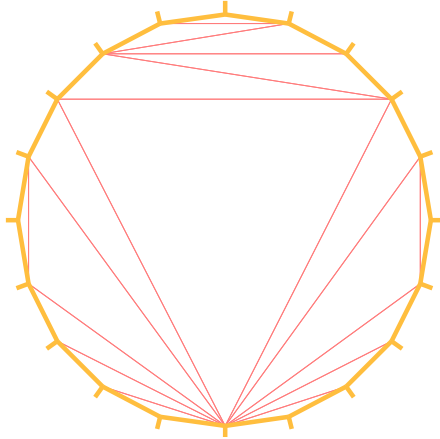


Figure 5.10: The empty mesh with boundary

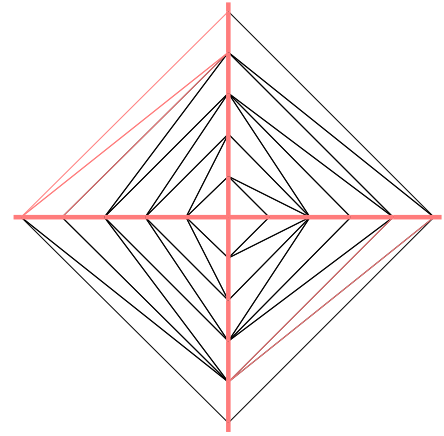


Figure 5.11: An empty mesh defined from a pseudo region numbering of triangle

5.3 Remeshing

5.3.1 Movemesh

Meshes can be translated, rotated and deformed by `movemesh`; this is useful for elasticity to watch the deformation due to the displacement $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y))$ of shape. It is also useful to handle free boundary problems or optimal shape problems.

If Ω is triangulated as $T_h(\Omega)$, and Φ is a displacement vector then $\Phi(T_h)$ is obtained by

```
mesh Th=movemesh(Th, [\Phi1, \Phi2]);
```

Sometimes the transformed mesh is invalid because some triangle have flip over (now has negative area). To spot such problems one may check the minimum triangle area in the transformed mesh with `checkmovemesh` before any real transformation.

Example 5.2 $\Phi_1(x, y) = x + k * \sin(y * \pi) / 10$, $\Phi_2(x, y) = y + k * \cos(y * \pi) / 10$ for a big number $k > 1$.

```
verbosity=4;
border a(t=0,1){x=t;y=0;label=1;};
border b(t=0,0.5){x=1;y=t;label=1;};
border c(t=0,0.5){x=1-t;y=0.5;label=1;};
border d(t=0.5,1){x=0.5;y=t;label=1;};
border e(t=0.5,1){x=1-t;y=1;label=1;};
border f(t=0,1){x=0;y=1-t;label=1;};
func uu= sin(y*pi)/10;
func vv= cos(x*pi)/10;

mesh Th = buildmesh ( a(6) + b(4) + c(4) +d(4) + e(4) + f(6));
plot(Th,wait=1,fill=1,ps="Lshape.eps"); // see figure 5.12
real coef=1;
real minT0= checkmovemesh(Th, [x,y]); // the min triangle area
```

```

while(1)                                // find a correct move mesh
{
    real minT=checkmovemesh(Th,[x+coef*uu,y+coef*vv]); // the min triangle area
    if (minT > minT0/5) break ;           // if big enough
    coef/=1.5;
}

Th=movemesh(Th,[x+coef*uu,y+coef*vv]);
plot(Th,wait=1,fill=1,ps="movemesh.eps"); // see figure 5.13

```

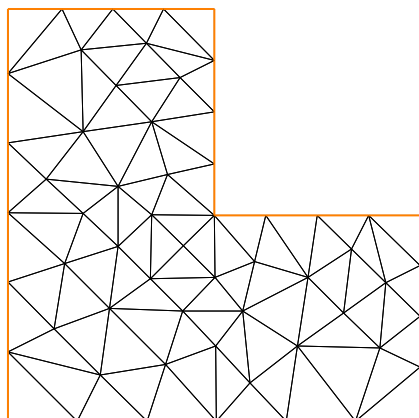


Figure 5.12: L-shape

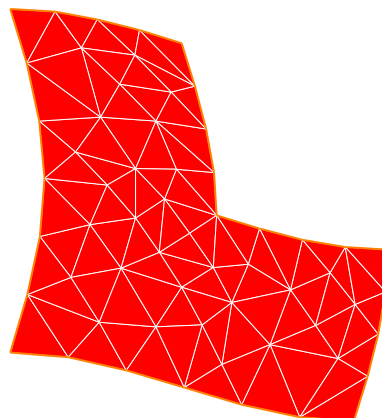


Figure 5.13: moved L-shape

Note 5.4 Consider a function u defined on a mesh Th . A statement like $Th=movemesh(Th\dots)$ does not change u and so the old mesh still exists. It will be destroyed when no function use it. A statement like $u = u$ redefines u on the new mesh Th with interpolation and therefore destroys the old Th if u was the only function using it.

Example 5.3 (movemesh.edp) Now, we given an example of moving mesh with a lagrangian function u defined on the moving mesh.

```

// simple movemesh example

mesh Th=square(10,10);
fespace Vh(Th,P1);
real t=0;

// ---
// the problem is how to build data without interpolation
// so the data u is moving with the mesh as you can see in the plot
// ---

Vh u=y;
for (int i=0;i<4;i++)
{
    t=i*0.1;
    Vh f= x*t;
    real minarea=checkmovemesh(Th,[x,y+f]);
    if (minarea >0 ) // movemesh will be ok

```



```

    Th=movemesh(Th,[x,y+f]);

cout << " Min area  " << minarea << endl;

real[int] tmp(u[].n);
tmp=u[];
u=0;
u[]=tmp;
plot(Th,u,wait=1);
};
//      In this program, since u is only defined on the last mesh, all the
//      previous meshes are deleted from memory.
//      -----

```

5.4 Regular Triangulation: hTriangle

For a set S , we define the diameter of S by

$$\text{diam}(S) = \sup\{|\mathbf{x} - \mathbf{y}|; \mathbf{x}, \mathbf{y} \in S\}$$

The sequence $\{\mathcal{T}_h\}_{h \downarrow 0}$ of Ω is called *regular* if they satisfy the following:

1.

$$\lim_{h \downarrow 0} \max\{\text{diam}(T_k) \mid T_k \in \mathcal{T}_h\} = 0$$

2. There is a number $\sigma > 0$ independent of h such that

$$\frac{\rho(T_k)}{\text{diam}(T_k)} \geq \sigma \quad \text{for all } T_k \in \mathcal{T}_h$$

where $\rho(T_k)$ are the diameter of the inscribed circle of T_k .

We put $h(\mathcal{T}_h) = \max\{\text{diam}(T_k) \mid T_k \in \mathcal{T}_h\}$, which is obtained by

```

mesh Th = .....;
fespace Ph(Th,P0);
Ph h = hTriangle;
cout << "size of mesh = " << h[].max << endl;

```

5.5 Adaptmesh

The function

$$f(x, y) = 10.0x^3 + y^3 + \tan^{-1}[\varepsilon/(\sin(5.0y) - 2.0x)] \quad \varepsilon = 0.0001$$

sharply varies in value and the initial mesh given by one of the commands of Section 5.1 cannot reflect its sharp variations.

Example 5.4

```

real eps = 0.0001;
real h=1;
real hmin=0.05;
func f = 10.0*x^3+y^3+h*atan2(eps,sin(5.0*y)-2.0*x);

mesh Th=square(5,5,[-1+2*x,-1+2*y]);
fespace Vh(Th,P1);
Vh fh=f;
plot(fh);
for (int i=0;i<2;i++)
{
    Th=adaptmesh(Th,fh);
    fh=f;
    plot(Th,fh,wait=1);
}

```

// old mesh is deleted

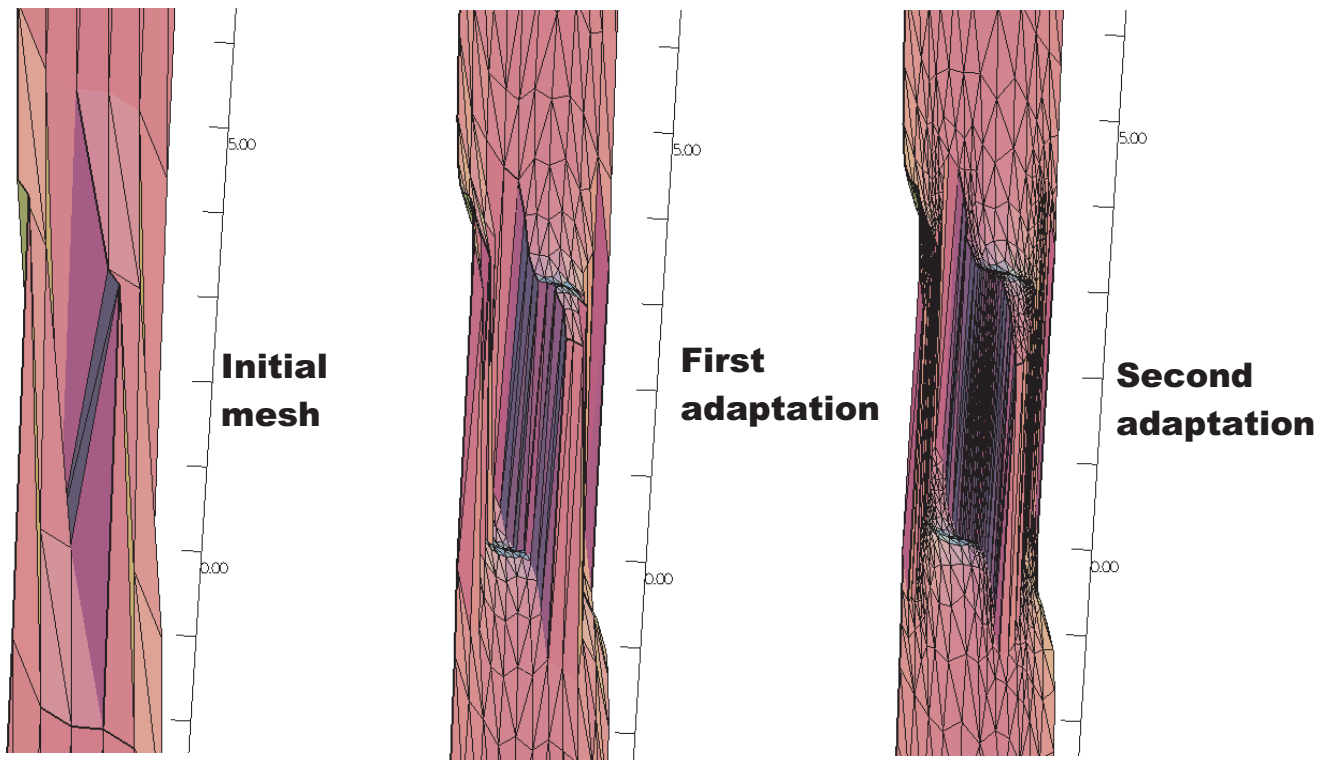


Figure 5.14: 3D graphs for the initial mesh and 1st and 2nd mesh adaptation

FreeFem++ uses a variable metric/Delaunay automatic meshing algorithm. The command

```
mesh ATh = adaptmesh(Th, f);
```

create the new mesh ATh adapted to the Hessian

$$D^2 f = (\partial^2 f / \partial x^2, \partial^2 f / \partial x \partial y, \partial^2 f / \partial y^2)$$

of a function (formula or FE-function). Mesh adaptation is a very powerful tool when the solution of a problem varies locally and sharply.

Here we solve the problem (2.1)-(2.2), when $f = 1$ and Ω is a L-shape domain.

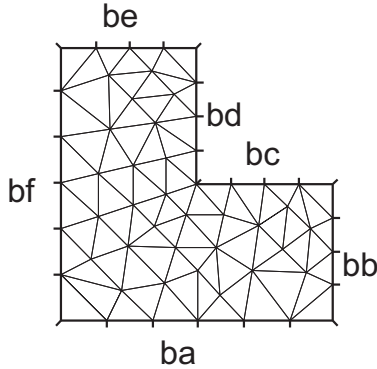


Figure 5.15: L-shape domain and its boundary name

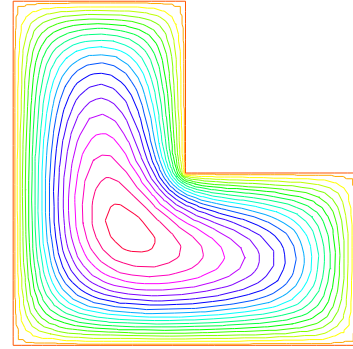


Figure 5.16: Final solution after 4-times adaptation

Example 5.5 (Adapt.edp) *The solution has the singularity $r^{3/2}$, $r = |x - \gamma|$ at the point γ of the intersection of two lines bc and bd (see Fig. 5.15).*

```

border ba(t=0,1.0){x=t; y=0; label=1;};
border bb(t=0,0.5){x=1; y=t; label=1;};
border bc(t=0,0.5){x=1-t; y=0.5; label=1;};
border bd(t=0.5,1){x=0.5; y=t; label=1;};
border be(t=0.5,1){x=1-t; y=1; label=1;};
border bf(t=0.0,1){x=0; y=1-t; label=1;};
mesh Th = buildmesh ( ba(6)+bb(4)+bc(4)+bd(4)+be(4)+bf(6) );
fespace Vh(Th,P1); // set FE space
Vh u,v; // set unknown and test function
func f = 1;
real error=0.1; // level of error
problem Poisson(u,v,solver=CG,eps=1.0e-6) =
    int2d(Th) ( dx(u)*dx(v) + dy(u)*dy(v) )
    - int2d(Th) ( f*v )
    + on(1,u=0) ;
for (int i=0;i< 4;i++)
{
    Poisson;
    Th=adaptmesh(Th,u,err=error);
    error = error/2;
} ;
plot(u);

```

To speed up the adaptation the default parameter `err` of `adaptmesh` is changed by hand; it specifies the required precision, so as to make the new mesh finer or coarser.

The problem is coercive and symmetric, so the linear system can be solved with the conjugate gradient method (parameter `solver=CG` with the stopping criteria on the residual, here

$\text{eps}=1.0\text{e-}6$). By `adaptmesh`, the slope of the final solution is correctly computed near the point of intersection of bc and bd as in Fig. 5.16.

This method is described in detail in [9]. It has a number of default parameters which can be modified :

Si $f1, f2$ sont des fonctions et `thold`, `Thnew` des maillages.

```
Thnew = adaptmesh(Thold, f1 ... );
Thnew = adaptmesh(Thold, f1,f2 ... );
Thnew = adaptmesh(Thold, [f1,f2] ... );
```

The additional parameters of `adaptmesh` not written here, hence the "..."

hmin= Minimum edge size. (`val` is a real. Its default is related to the size of the domain to be meshed and the precision of the mesh generator).

hmax= Maximum edge size. (`val` is a real. It defaults to the diameter of the domain to be meshed)

err= P_1 interpolation error level (0.01 is the default).

errg= Relative geometrical error. By default this error is 0.01, and in any case it must be lower than $1/\sqrt{2}$. Meshes created with this option may have some edges smaller than the `-hmin` due to geometrical constraints.

nbvx= Maximum number of vertices generated by the mesh generator (9000 is the default).

nbsmooth= number of iterations of the smoothing procedure (5 is the default).

nbjacob= number of iterations in a smoothing procedure during the metric construction, 0 means no smoothing (6 is the default).

ratio= ratio for a prescribed smoothing on the metric. If the value is 0 or less than 1.1 no smoothing is done on the metric (1.8 is the default).

If $\text{ratio} > 1.1$, the speed of mesh size variations is bounded by $\log(\text{ratio})$. Note: As ratio gets closer to 1, the number of generated vertices increases. This may be useful to control the thickness of refined regions near shocks or boundary layers .

omega= relaxation parameter for the smoothing procedure (1.0 is the default).

iso= If true, forces the metric to be isotropic (false is the default).

absererror= If false, the metric is evaluated using the criterium of equi-repartition of relative error (false is the default). In this case the metric is defined by

$$\mathcal{M} = \left(\frac{1}{\text{err coef}^2} \frac{|\mathcal{H}|}{\max(\text{CutOff}, |\eta|)} \right)^p \quad (5.1)$$

otherwise, the metric is evaluated using the criterium of equi-distribution of errors. In this case the metric is defined by

$$\mathcal{M} = \left(\frac{1}{\text{err coef}^2} \frac{|\mathcal{H}|}{\sup(\eta) - \inf(\eta)} \right)^p. \quad (5.2)$$

cutoff= lower limit for the relative error evaluation (1.0e-6 is the default).

verbosity= informational messages level (can be chosen between 0 and ∞). Also changes the value of the global variable `verbosity` (obsolete).

inquire= To inquire graphically about the mesh (false is the default).

splitpbedge= If true, splits all internal edges in half with two boundary vertices (true is the default).

maxsubdiv= Changes the metric such that the maximum subdivision of a background edge is bound by `val` (always limited by 10, and 10 is also the default).

rescaling= if true, the function with respect to which the mesh is adapted is rescaled to be between 0 and 1 (true is the default).

keepbackvertices= if true, tries to keep as many vertices from the original mesh as possible (true is the default).

isMetric= if true, the metric is defined explicitly (false is the default). If the 3 functions m_{11}, m_{12}, m_{22} are given, they directly define a symmetric matrix field whose Hessian is computed to define a metric. If only one function is given, then it represents the isotropic mesh size at every point.

For example, if the partial derivatives $f_{xx} (= \partial^2 f / \partial x^2)$, $f_{xy} (= \partial^2 f / \partial x \partial y)$, $f_{yy} (= \partial^2 f / \partial y^2)$ are given, we can set

```
Th=adaptmesh(Th,fxx,fxy,fyy,IsMetric=1,nbvx=10000,hmin=hmin);
```

power= exponent power of the Hessian used to compute the metric (1 is the default).

thetamax= minimum corner angle of in degrees (default is 10°) where the corner is ABC and the angle is the angle of the two vectors AB, BC , (0 imply no corner, 90 imply perp. corner , ...).

splitin2= boolean value. If true, splits all triangles of the final mesh into 4 sub-triangles.

metric= an array of 3 real arrays to set or get metric data information. The size of these three arrays must be the number of vertices. So if m_{11}, m_{12}, m_{22} are three P1 finite elements related to the mesh to adapt, you can write: `metric=[m11[],m12[],m22[]]` (see file `convect-apt.edp` for a full example)

nomeshgeneration= If true, no adapted mesh is generated (useful to compute only a metric).

periodic= Writing `periodic=[[4,y],[2,y],[1,x],[3,x]]`; builds an adapted periodic mesh. The sample build a bi-periodic mesh of a square. (see `periodic finite element spaces 6`, and see `sphere.edp` for a full example)

We can use the command `adaptmesh` to build uniform mesh with a constant mesh size. So to build a mesh with a constant mesh size equal to $\frac{1}{30}$ try:

Example 5.6 *uniformmesh.edp*

```

mesh Th=square(2,2); // to have initial mesh
plot(Th,wait=1,ps="square-0.eps");
Th= adaptmesh(Th,1./3As writing //
0.,IsMetric=1,nbv=10000); //
plot(Th,wait=1,ps="square-1.eps");
Th= adaptmesh(Th,1./30.,IsMetric=1,nbv=10000); // more the one time du to
Th= adaptmesh(Th,1./30.,IsMetric=1,nbv=10000); // adaptation bound
maxsubdiv=
plot(Th,wait=1,ps="square-2.eps");

```

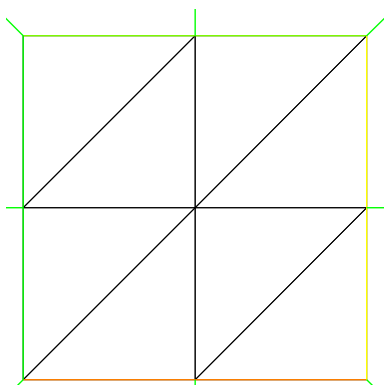


Figure 5.17: Initial mesh

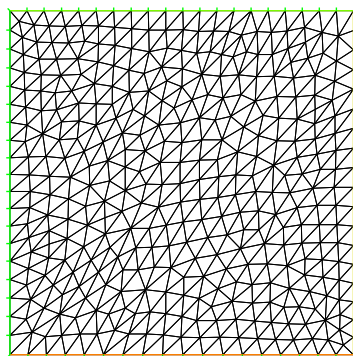


Figure 5.18: first iteration

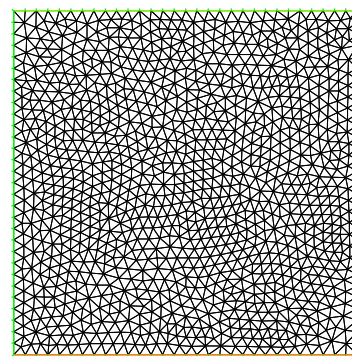


Figure 5.19: last iteration

5.6 Trunc

Two operators have been introduced to remove triangles from a mesh or to divide them. Operator `trunc` has two parameters

label= sets the label number of new boundary item (one by default)

split= sets the level n of triangle splitting. each triangle is splitted in $n \times n$ (one by default).

To create the mesh `Th3` where all triangles of a mesh `Th` are splitted in 3×3 , just write:

```

mesh Th3 = trunc(Th,1,split=3);

```

The `truncmesh.edp` example construct all "trunc" mesh to the support of the basic function of the space V_h (cf. $\text{abs}(u) > 0$), split all the triangles in 5×5 , and put a label number to 2 on new boundary.

```

mesh Th=square(3,3);
fespace Vh(Th,P1);
Vh u;
int i,n=u.n;
u=0;
for (i=0;i<n;i++)                                // all degree of freedom
{
    u[][i]=1;                                       // the basic function i
    plot (u,wait=1);
    mesh Sh1=trunc(Th,abs(u)>1.e-10,split=5,label=2);
    plot (Th,Sh1,wait=1,ps="trunc"+i+".eps");        // plot the mesh of
                                                    // the function's support
    u[][i]=0;                                       // reset
}

```

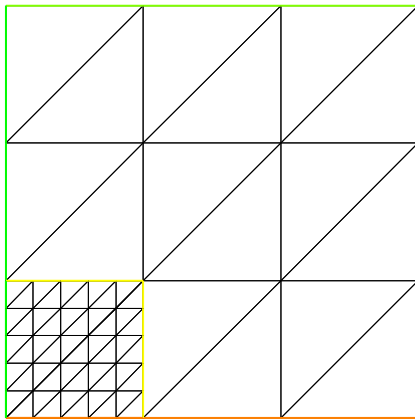


Figure 5.20: mesh of support the function P_1 number 0, splitted in 5×5

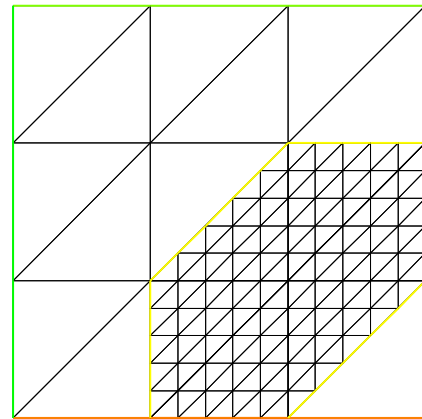


Figure 5.21: mesh of support the function P_1 number 6, splitted in 5×5

5.7 Splitmesh

Another way to split mesh triangles is to use `splitmesh`, for example:

```

{
    // new stuff 2004 splitmesh (version 1.37)
    assert (version>=1.37);
    border a(t=0,2*pi){ x=cos(t); y=sin(t);label=1;}
    mesh Th=buildmesh(a(20));
    plot (Th,wait=1,ps="nosplitmesh.eps");           // see figure 5.22
    Th=splitmesh(Th,1+5*(square(x-0.5)+y*y));
    plot (Th,wait=1,ps="splitmesh.eps");           // see figure 5.23
}

```

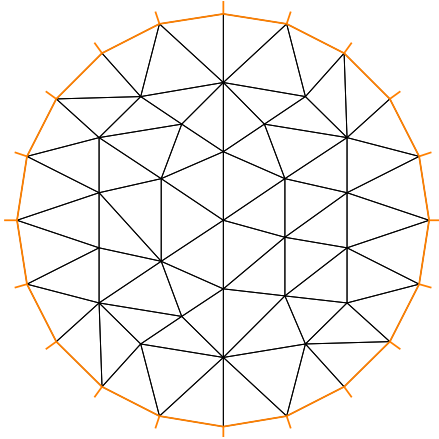


Figure 5.22: initial mesh

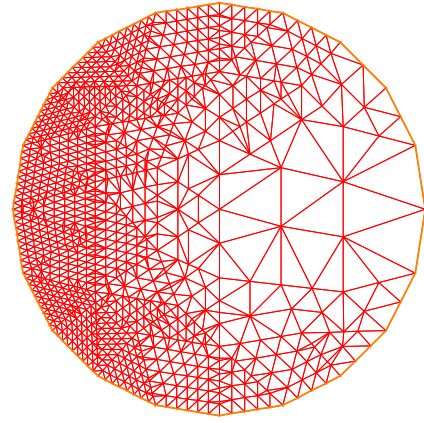


Figure 5.23: all left mesh
 triangle is split conformally in
 $\text{int}(1+5 \cdot (\text{square}(x-0.5)+y \cdot y)^2$
 triangles.

5.8 Meshing Examples

Example 5.7 (Two rectangles touching by a side)

```

border a(t=0,1){x=t;y=0;};
border b(t=0,1){x=1;y=t;};
border c(t=1,0){x=t;y=1;};
border d(t=1,0){x=0;y=t;};
border c1(t=0,1){x=t;y=1;};
border e(t=0,0.2){x=1;y=1+t;};
border f(t=1,0){x=t;y=1.2;};
border g(t=0.2,0){x=0;y=1+t;};
int n=1;
mesh th = buildmesh(a(10*n)+b(10*n)+c(10*n)+d(10*n));
mesh TH = buildmesh(c1(10*n) + e(5*n) + f(10*n) + g(5*n));
plot(th,TH,ps="TouchSide.esp");
```

// Fig. 5.24

Example 5.8 (NACA0012 Airfoil)

```

border upper(t=0,1) { x = t;
  y = 0.17735*sqrt(t)-0.075597*t
  - 0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4); }
border lower(t=1,0) { x = t;
  y= -(0.17735*sqrt(t)-0.075597*t
  -0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4)); }
border c(t=0,2*pi) { x=0.8*cos(t)+0.5; y=0.8*sin(t); }
mesh Th = buildmesh(c(30)+upper(35)+lower(35));
plot(Th,ps="NACA0012.eps",bw=1);
```

// Fig. 5.25

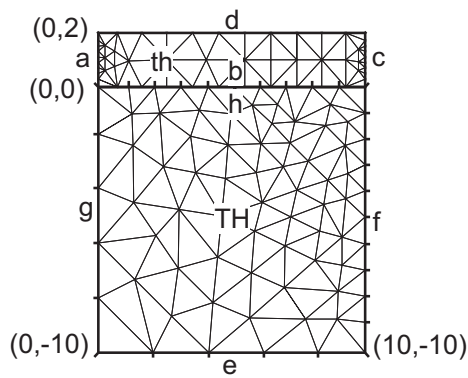


Figure 5.24: Two rectangles touching by a side

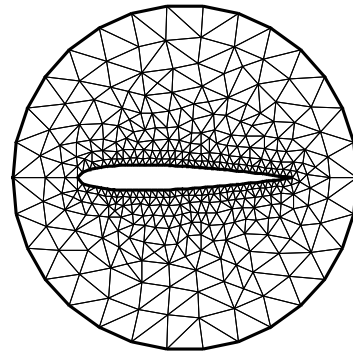


Figure 5.25: NACA0012 Airfoil

Example 5.9 (Cardioid)

```

real b = 1, a = b;
border C(t=0,2*pi) { x=(a+b)*cos(t)-b*cos((a+b)*t/b);
                      y=(a+b)*sin(t)-b*sin((a+b)*t/b); }
mesh Th = buildmesh(C(50));
plot(Th,ps="Cardioid.eps",bw=1); // Fig. 5.26

```

Example 5.10 (Cassini Egg)

```

border C(t=0,2*pi) { x=(2*cos(2*t)+3)*cos(t);
                      y=(2*cos(2*t)+3)*sin(t); }
mesh Th = buildmesh(C(50));
plot(Th,ps="Cassini.eps",bw=1); // Fig. 5.27

```

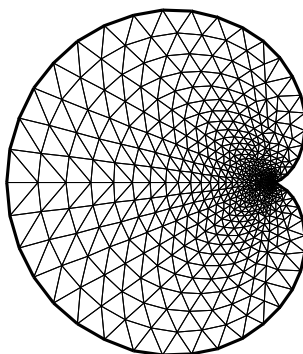


Figure 5.26: Domain with Cardioid curve boundary

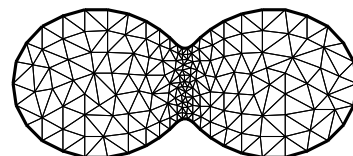


Figure 5.27: Domain with Cassini Egg curve boundary

Example 5.11 (By cubic Bezier curve)

```

//      A cubic Bezier curve connecting two points with two control points
func real bzi(real p0, real p1, real q1, real q2, real t)
{
    return p0*(1-t)^3+q1*3*(1-t)^2*t+q2*3*(1-t)*t^2+p1*t^3;
}

real[int] p00=[0,1], p01=[0,-1], q00=[-2,0.1], q01=[-2,-0.5];
real[int] p11=[1,-0.9], q10=[0.1,-0.95], q11=[0.5,-1];
real[int] p21=[2,0.7], q20=[3,-0.4], q21=[4,0.5];
real[int] q30=[0.5,1.1], q31=[1.5,1.2];
border G1(t=0,1) { x=bzi(p00[0],p01[0],q00[0],q01[0],t);
                  y=bzi(p00[1],p01[1],q00[1],q01[1],t); }
border G2(t=0,1) { x=bzi(p01[0],p11[0],q10[0],q11[0],t);
                  y=bzi(p01[1],p11[1],q10[1],q11[1],t); }
border G3(t=0,1) { x=bzi(p11[0],p21[0],q20[0],q21[0],t);
                  y=bzi(p11[1],p21[1],q20[1],q21[1],t); }
border G4(t=0,1) { x=bzi(p21[0],p00[0],q30[0],q31[0],t);
                  y=bzi(p21[1],p00[1],q30[1],q31[1],t); }

int m=5;
mesh Th = buildmesh (G1 (2*m)+G2 (m)+G3 (3*m)+G4 (m));
plot (Th,ps="Bezier.eps",bw=1);

```

// Fig 5.28

Example 5.12 (Section of Engine)

```

real a= 6., b= 1., c=0.5;
border L1(t=0,1) { x= -a; y= 1+b - 2*(1+b)*t; }
border L2(t=0,1) { x= -a+2*a*t; y= -1-b*(x/a)*(x/a)*(3-2*abs(x)/a ); }
border L3(t=0,1) { x= a; y=-1-b + (1+ b )*t; }
border L4(t=0,1) { x= a - a*t; y=0; }
border L5(t=0,pi) { x= -c*sin(t)/2; y=c/2-c*cos(t)/2; }
border L6(t=0,1) { x= a*t; y=c; }
border L7(t=0,1) { x= a; y=c + (1+ b-c )*t; }
border L8(t=0,1) { x= a-2*a*t; y= 1+b*(x/a)*(x/a)*(3-2*abs(x)/a ); }
mesh Th = buildmesh (L1 (8)+L2 (26)+L3 (8)+L4 (20)+L5 (8)+L6 (30)+L7 (8)+L8 (30));
plot (Th,ps="Engine.eps",bw=1);

```

// Fig. 5.29

Example 5.13 (Domain with U-shape channel)

```

real d = 0.1;
border L1(t=0,1-d) { x=-1; y=-d-t; }
border L2(t=0,1-d) { x=-1; y=1-t; }
border B(t=0,2) { x=-1+t; y=-1; }
border C1(t=0,1) { x=t-1; y=d; }
border C2(t=0,2*d) { x=0; y=d-t; }
border C3(t=0,1) { x=-t; y=-d; }
border R(t=0,2) { x=1; y=-1+t; }
border T(t=0,2) { x=1-t; y=1; }
int n = 5;
mesh Th = buildmesh (L1 (n/2)+L2 (n/2)+B (n)+C1 (n)+C2 (3)+C3 (n)+R (n)+T (n));
plot (Th,ps="U-shape.eps",bw=1);

```

// width of U-shape Fig 5.30

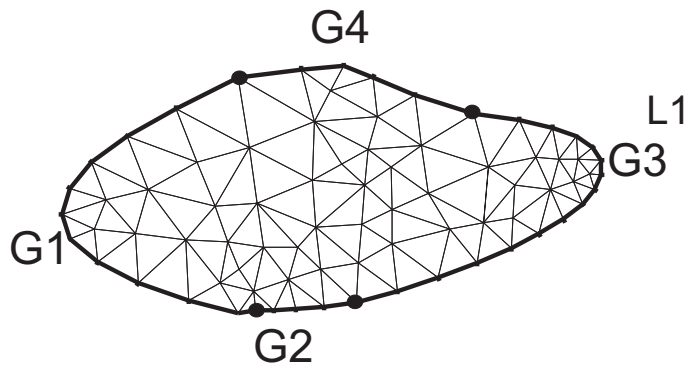


Figure 5.28: Boundary drawn by Bezier curves

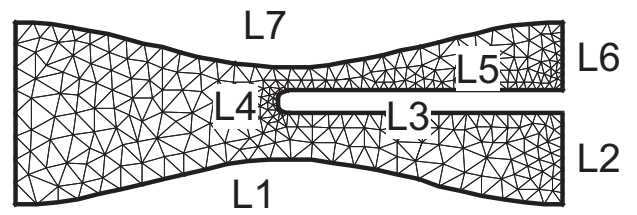


Figure 5.29: Section of Engine

Example 5.14 (Domain with V-shape cut)

```

real dAg = 0.01; // angle of V-shape
border C(t=dAg,2*pi-dAg) { x=cos(t); y=sin(t); };
real[int] pa(2), pb(2), pc(2);
pa[0] = cos(dAg); pa[1] = sin(dAg);
pb[0] = cos(2*pi-dAg); pb[1] = sin(2*pi-dAg);
pc[0] = 0; pc[1] = 0;
border seg1(t=0,1) { x=(1-t)*pb[0]+t*pc[0]; y=(1-t)*pb[1]+t*pc[1]; };
border seg2(t=0,1) { x=(1-t)*pc[0]+t*pa[0]; y=(1-t)*pc[1]+t*pa[1]; };
mesh Th = buildmesh(seg1(20)+C(40)+seg2(20));
plot(Th,ps="V-shape.eps",bw=1); // Fig. 5.31

```

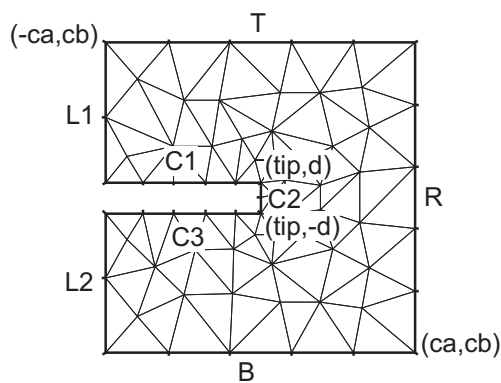


Figure 5.30: Domain with U-shape channel changed by d

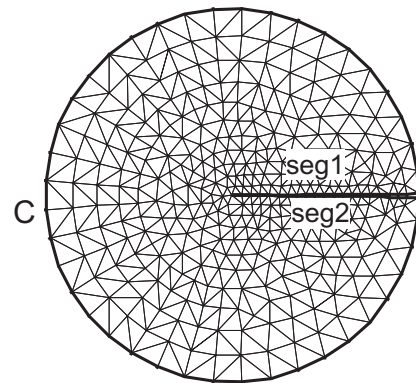


Figure 5.31: Domain with V-shape cut changed by dAg

Example 5.15 (Smiling face)

```

real d=0.1;
int m=5;
real a=1.5, b=2, c=0.7, e=0.01;
border F(t=0,2*pi) { x=a*cos(t); y=b*sin(t); }
border E1(t=0,2*pi) { x=0.2*cos(t)-0.5; y=0.2*sin(t)+0.5; }
border E2(t=0,2*pi) { x=0.2*cos(t)+0.5; y=0.2*sin(t)+0.5; }
func real st(real t) {
    return sin(pi*t)-pi/2;
}
border C1(t=-0.5,0.5) { x=(1-d)*c*cos(st(t)); y=(1-d)*c*sin(st(t)); }
border C2(t=0,1){x=((1-d)+d*t)*c*cos(st(0.5));y=((1-d)+d*t)*c*sin(st(0.5));}
border C3(t=0.5,-0.5) { x=c*cos(st(t)); y=c*sin(st(t)); }
border C4(t=0,1) { x=(1-d*t)*c*cos(st(-0.5)); y=(1-d*t)*c*sin(st(-0.5));}

border C0(t=0,2*pi) { x=0.1*cos(t); y=0.1*sin(t); }
mesh Th=buildmesh(F(10*m)+C1(2*m)+C2(3)+C3(2*m)+C4(3)
                  +C0(m)+E1(-2*m)+E2(-2*m));
plot(Th,ps="SmileFace.eps",bw=1); // see Fig. 5.32

```

Example 5.16 (3point bending)

```

// Square for Three-Point Bend Specimens fixed on Fix1, Fix2
// It will be loaded on Load.

real a=1, b=5, c=0.1;
int n=5, m=b*n;
border Left(t=0,2*a) { x=-b; y=a-t; }
border Bot1(t=0,b/2-c) { x=-b+t; y=-a; }
border Fix1(t=0,2*c) { x=-b/2-c+t; y=-a; }
border Bot2(t=0,b-2*c) { x=-b/2+c+t; y=-a; }
border Fix2(t=0,2*c) { x=b/2-c+t; y=-a; }
border Bot3(t=0,b/2-c) { x=b/2+c+t; y=-a; }
border Right(t=0,2*a) { x=b; y=-a+t; }
border Top1(t=0,b-c) { x=b-t; y=a; }
border Load(t=0,2*c) { x=c-t; y=a; }
border Top2(t=0,b-c) { x=-c-t; y=a; }
mesh Th = buildmesh(Left(n)+Bot1(m/4)+Fix1(5)+Bot2(m/2)+Fix2(5)+Bot3(m/4)
                    +Right(n)+Top1(m/2)+Load(10)+Top2(m/2));
plot(Th,ps="ThreePoint.eps",bw=1); // Fig. 5.33

```

5.9 How to change the label of elements and border elements of a mesh

Changing the label of elements and border elements will be done using the keyword **change**. The parameters for this command line are for a two dimensional and dimensional case:

label = is a vector of integer that contains successive pair of the old label number to the new label number .

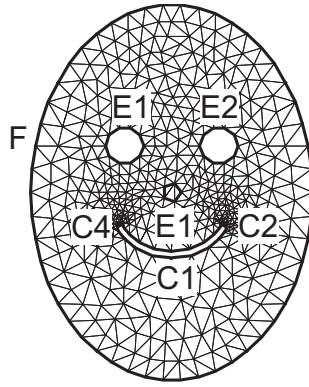


Figure 5.32: Smiling face (Mouth is changeable)

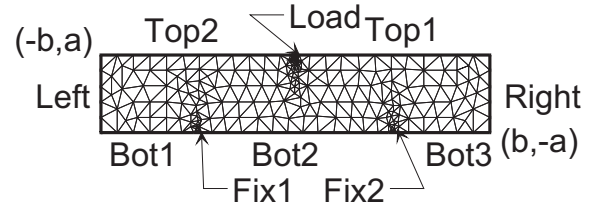


Figure 5.33: Domain for three-point bending test

region = is a vector of integer that contains successive pair of the old region number to new region number.

label = is a integer function with given the new value of the label (version 3.21).

fregion = is a integer function with given the new value of the region .

These vectors are composed of n_l successive pair of number O, N where n_l is the number (label or region) that we want to change. For example, we have

$$\text{label} = [O_1, N_1, \dots, O_{n_l}, N_{n_l}] \quad (5.3)$$

$$\text{region} = [O_1, N_1, \dots, O_{n_l}, N_{n_l}] \quad (5.4)$$

An example of using this function is given in "glumesh2D.edp":

Example 5.17 (glumesh2D.edp)

```

1:
2: mesh Th1=square(10,10);
3: mesh Th2=square(20,10,[x+1,y]);
4: verbosity=3;
5: int[int] r1=[2,0], r2=[4,0];
6: plot(Th1,wait=1);
7: Th1=change(Th1,label=r1); // Change the label of Edges 2 in 0.
8: plot(Th1,wait=1);
9: Th2=change(Th2,label=r2); // Change the label of Edges 4 in 0.
10: mesh Th=Th1+Th2; // 'gluing together' of meshes Th1 and Th2
11: cout << " nb lab = " << int1d(Th1,1,3,4)(1./lenEdge)+int1d(Th2,1,2,3)(1./lenEdge)
12: << " == " << int1d(Th,1,2,3,4)(1./lenEdge) << " == " << ((10+20)+10)*2
<< endl;
13: plot(Th,wait=1);
14: fespace Vh(Th,P1);
15: macro Grad(u) [dx(u),dy(u)]; // definition of a macro
16: Vh u,v;
17: solve P(u,v)=int2d(Th)(Grad(u)'*Grad(v))-int2d(Th)(v)+on(1,3,u=0);
18: plot(u,wait=1);

```

“gluing” different mesh In line 10 of previous file, the method to “gluing” different mesh of the same dimension in FreeFem++ is using. This function is the operator “+” between meshes. The method implemented need that the point in adjacent mesh are the same.

5.10 Mesh in three dimensions

5.10.1 Read/Write Statements for a Mesh in 3D

In three dimensions, the file mesh format supported for input and output files by FreeFem++ are the extension .msh and .mesh. These formats are described in the chapter on Mesh Files in two dimensions.

extension file .msh The structure of the files with extension .msh in 3D is given in Table 5.2. In this structure, n_v denotes the number of vertices, n_{tet} the number of tetrahedra and n_{tri} the number of triangles. For each vertex q^i , $i = 1, \dots, n_v$, we denote by (q_x^i, q_y^i, q_z^i) the x -coordinate, the y -coordinate and the z -coordinate. Each tetrahedra T_k , $k = 1, \dots, n_{tet}$ has four vertices $q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}$. The boundary consists of an union of triangles. Each triangle be_j , $j = 1, \dots, n_{tri}$ has three vertices $q^{j_1}, q^{j_2}, q^{j_3}$.

n_v	n_{tet}	n_{tri}		
q_x^1	q_y^1	q_z^1	Vertex label	
q_x^2	q_y^2	q_z^2	Vertex label	
\vdots	\vdots	\vdots	\vdots	
$q_x^{n_v}$	$q_y^{n_v}$	$q_z^{n_v}$	Vertex label	
1 ₁	1 ₂	1 ₃	1 ₄	region label
2 ₁	2 ₂	2 ₃	2 ₄	region label
\vdots	\vdots	\vdots	\vdots	\vdots
$(n_{tet})_1$	$(n_{tet})_2$	$(n_{tet})_3$	$(n_{tet})_4$	region label
1 ₁	1 ₂	1 ₃	boundary label	
2 ₁	2 ₂	2 ₃	boundary label	
\vdots	\vdots	\vdots	\vdots	
$(n_{tri})_1$	$(n_{tri})_2$	$(n_{tri})_3$	boundary label	

Table 5.2: The structure of mesh file format “.msh” in three dimensions.

extension file .mesh The data structure for a three dimensional mesh is composed of the data structure presented in Section 12.1 and a data structure for tetrahedra. The tetrahedra of a three dimensional mesh are refereed using the following field:

- Tetrahedra
 $(\text{I}) \text{ NbOfTetrahedrons}$
 $\left(\left(@@\text{Vertex}_i^j, j=1,4 \right), (\text{I}) \text{ Ref}\phi_i^{tet}, i=1, \text{NbOfTetrahedrons} \right)$

This field is express with the notation of Section 12.1.

5.10.2 TeGen: A tetrahedral mesh generator

TetGen TetGen is a software developed by Dr. Hang Si of Weierstrass Institute for Applied Analysis and Stochastics of Berlin in Germany [36]. TetGen is a free for research and non-commercial uses. For any commercial licence utilization, a commercial licence is available upon request to Hang Si.

This software is a tetrahedral mesh generator of a three dimensional domain defined by its boundary. The input domain take into account a polyhedral or a piecewise linear complex. This tetrahedralization is a constrained Delaunay tetrahedralization.

The method used in TetGen to control the quality of the mesh is a Delaunay refinement due to Shewchuk [37]. The quality measure of this algorithm is the Radius-Edge Ratio (see Section 1.3.1 [36] for more details). A theoretical bounds of this ratio of the algorithm of Shewchuk is obtained for a given complex of vertices, constrained segments and facets of surface mesh, with no input angle less than 90 degree. This theoretical bounds is 2.0.

The launch of Tetgen is done with the keyword `tetg`. The parameters of this command line is:

label = is a vector of integer that contains the old labels number at index $2i$ and the new labels number at index $2i + 1$ of Triangles. This parameters is initialized as label for the keyword change (5.3).

switch = A string expression. This string corresponds to the command line switch of Tetgen see Section 3.2 of [36].

nbofholes= Number of holes (default value size of `holelist/3` (version 3.11)).

holelist = This array correspond to **holelist** of tetgenio data structure [36]. A real vector of size $3 \times \text{nbofholes}$. In TetGen, each hole is associated with a point inside this domain. This vector is $x_1^h, y_1^h, z_1^h, x_2^h, y_2^h, z_2^h, \dots$, where x_i^h, y_i^h, z_i^h is the associated point with the i^{th} hole.

nbofregions = Number of regions (size of `regionlist/5` (version 3.11)).

regionlist = This array corresponds to **regionlist** of tetgenio data structure [36]. The attribute and the volume constraint of region are given in this real vector of size $5 \times \text{nbofregions}$. The i^{th} region is described by five elements: x -coordinate, y -coordinate and z -coordinate of a point inside this domain (x_i, y_i, z_i); the attribute (at_i) and the maximum volume for tetrahedra ($mvol_i$) for this region. The `regionlist` vector is: $x_1, y_1, z_1, at_1, mvol_1, x_2, y_2, z_2, at_2, mvol_2, \dots$.

nboffacetcl= Number of facets constraints size of `facetcl/2` (version 3.11)).

facetcl= This array corresponds to **facetconstraintlist** of tetgenio data structure [36]. The i^{th} facet constraint is defined by the facet marker Ref_i^{fc} and the maximum area for faces $marea_i^{fc}$. The `facetcl` array is: $Ref_1^{fc}, marea_1^{fc}, Ref_2^{fc}, marea_2^{fc}, \dots$. This parameters has no effect if switch `q` is not selected.

Principal switch parameters in TetGen:

`p` Tetrahedralization of boundary.

`q` Quality mesh generation. The bound of Radius-Edge Ratio will be given after the option `q`. By default, this value is 2.0.

- a Construct with the volumes constraints on tetrahedra. These volumes constraints are defined with the bound of the previous switch `q` or in the parameter `regionlist`.
- A Attributes reference to region given in the `regionlist`. The other regions have label 0. The option `AA` gives a different label at each region. This switch work with the option '`p`'. If option '`r`' is used, this switch has no effect.
- r Reconstructs and Refines a previously generated mesh. This character is only used with the command line `tetgreconstruction`.
- Y This switch allow to preserve the mesh on the exterior boundary. This switch must be used to ensure conformal mesh between two adjacents mesh.
- YY This switch allow to preserve the mesh on the exterior and interior boundary.
- C The consistency of the result's mesh is testing by TetGen.
- CC The consistency of the result's mesh is testing by TetGen and also checks constrained delaunay mesh (if '`p`' switch is selected) or the consistency of Conformal Delaunay (if '`q`' switch is selected).
- V Give information of the work of TetGen. More information can be obtained in specified '`VV`' or '`VVV`'.
- Q Quiet: No terminal output except errors
- M The coplanar facets are not merging.
- T Set a tolerance for coplanar test. The default value is $1e - 8$.
- d Itersections of facets are detected.

To obtain a tetrahedral mesh generator with `tetgen`, we need the surface mesh of three dimensional domain. We give now the command line in `FreeFem++` to construct these meshes.

keyword: “`movemesh23`” A simple method to construct a surface is to place a two dimensional domain in a three dimensional space. This corresponding to move the domain by a displacement vector of this form $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), \Phi_3(x, y))$. The result of moving a two dimensional mesh `Th2` by this three dimensional displacement is obtained using:

```
mesh3 Th3 = movemesh23 (Th2,transfo=[ $\Phi_1, \Phi_2, \Phi_3$ ]);
```

The parameters of this command line are:

transfo = [Φ_1, Φ_2, Φ_3] set the displacement vector of transformation $\Phi(x, y) = [\Phi_1(x, y), \Phi_2(x, y), \Phi_3(x, y)]$

label = set integer label of triangles

orientation= set integer orientation of mesh.

ptmerge = A real expression. When you transform a mesh, some points can be merged. This parameters is the criteria to define two merging points. By default, we use

$$ptmerge = 1e - 7 \text{ Vol}(B),$$

where B is the smallest axis parallel boxes containing the discretized domain of Ω and $\text{Vol}(B)$ is the volume of this box.

We can ‘do a ‘gluing’ of surface meshes using the process given in Section 5.9. An example to obtain a three dimensional mesh using the command line `tetg` and `movemesh23` is given in the file `tetgencube.edp`.

Example 5.18 (tetgencube.edp)

```

// file tetgencube.edp

load "msh3"
load "tetgen"

real x0,x1,y0,y1;
x0=1.; x1=2.; y0=0.; y1=2*pi;
mesh Thsq1 = square(5,35,[x0+(x1-x0)*x,y0+(y1-y0)*y]);

func ZZ1min = 0;
func ZZ1max = 1.5;
func XX1 = x;
func YY1 = y;

mesh3 Th31h = movemesh23(Thsq1,transfo=[XX1,YY1,ZZ1max]);
mesh3 Th31b = movemesh23(Thsq1,transfo=[XX1,YY1,ZZ1min]);

// //////////////////////////////////////

x0=1.; x1=2.; y0=0.; y1=1.5;
mesh Thsq2 = square(5,8,[x0+(x1-x0)*x,y0+(y1-y0)*y]);

func ZZ2 = y;
func XX2 = x;
func YY2min = 0.;
func YY2max = 2*pi;

mesh3 Th32h = movemesh23(Thsq2,transfo=[XX2,YY2max,ZZ2]);
mesh3 Th32b = movemesh23(Thsq2,transfo=[XX2,YY2min,ZZ2]);

// //////////////////////////////////////

x0=0.; x1=2*pi; y0=0.; y1=1.5;
mesh Thsq3 = square(35,8,[x0+(x1-x0)*x,y0+(y1-y0)*y]);
func XX3min = 1.;
func XX3max = 2.;
func YY3 = x;
func ZZ3 = y;

mesh3 Th33h = movemesh23(Thsq3,transfo=[XX3max,YY3,ZZ3]);
mesh3 Th33b = movemesh23(Thsq3,transfo=[XX3min,YY3,ZZ3]);

// //////////////////////////////////////
mesh3 Th33 = Th31h+Th31b+Th32h+Th32b+Th33h+Th33b; // "gluing" surface meshes
to obtain the surface of cube
savemesh(Th33,"Th33.mesh");

// build a mesh of a axis parallel box with TetGen
real[int] domain =[1.5,pi,0.75,145,0.0025];
mesh3 Thfinal = tetg(Th33,switch="paAAQY",regionlist=domain); //
Tetrahelize the interior of the cube with tetgen
savemesh(Thfinal,"Thfinal.mesh");

```

```

//      build a mesh of a half cylindrical shell of interior radius 1. and
//      exterior radius 2 and heigh 1.5
func mv2x = x*cos(y);
func mv2y = x*sin(y);
func mv2z = z;
mesh3 Thmv2 = movemesh3(Thfinal, transfo=[mv2x,mv2y,mv2z]);
savemesh(Thmv2,"halfcylindricalshell.mesh")

```

The command `movemesh` is describe in the following section.

The keyword “tetgtransfo” This keyword correspond to a composition of command line `tetg` and `movemesh23`:

```
tetgtransfo( Th2, transfo= [Φ1, Φ2, Φ3] ), ... ) = tetg( Th3surf, ... ),
```

where `Th3surf = movemesh23(Th2,tranfo=[Φ1, Φ2, Φ3])` and `Th2` is the input two dimensional mesh of `tetgtransfo`.

The parameters of this command line are on the one hand the parameters:

`label, switch, regionlist nboffacetcl facetcl`

of keyword `tetg` and on the other hand the parameter `ptmerge` of keyword `movemesh23`.

Remark: To use `tetgtransfo`, the result’s mesh of `movemesh23` must be an closed surface and define one region only. Therefore, the parameter `regionlist` is defined for one region. An example of this keyword can be found in line of file “`buildlayers.edp`”

The keyword ”tetgconvexhull” `FreeFem++` , using `tetgen`, is able to build a tetrahedralization from a set of points. This tetrahedralization is a Delaunay mesh of the convex hull of the set of points.

The coordinates of the points can be initialized in two ways. The first is a file that contains the coordinate of points $X_i = (x_i, y_i, z_i)$. This files is organized as follows:

$$\begin{array}{ccc}
 n_v & & \\
 x_1 & y_1 & z_1 \\
 x_2 & y_2 & z_2 \\
 \vdots & \vdots & \vdots \\
 x_{n_v} & y_{n_v} & z_{n_v}
 \end{array}$$

The second way is to give three arrays that correspond respectively to the x –coordinates, y –coordinates and z –coordinates.

The parameters of this command line are

switch = A string expression. This string corresponds to the command line *switch* of `TetGen` see Section 3.2 of [36].

reftet = An integer expression. set the label of tetrahedra.

label = An integer expression. set the label of triangles.

In the string `switch`, we can’t used the option ‘p’ and ‘q’ of `tetgen`.

5.10.3 Reconstruct/Refine a three dimensional mesh with TetGen

Mesheres in three dimension can be refined using TetGen with the command line `tetgreconstruction`. The parameter of this keyword are

region= an integer array that allow to change the region number of tetrahedra. This array is defined as the parameter `reftet` in the keyword `change`.

label= an integer array that allow to change the label of boundary triangles. This array is defined as the parameter `label` in the keyword `change`.

sizevolume= a reel function. This function allows to constraint volume size of tetrahedra in the domain. (see example 5.31 to build 3d adapt mesh)

The parameter switch `nbofregions`, `regionlist`, `nboffacetcl` and `facetcl` of the command line which call TetGen (`tetg`) is used for `tetgrefine`.

In the parameter `switch=`, the character 'r' should be used without the character 'p'. For instance, see the manual of TetGen [36] for effect of 'r' to other character.

The parameter `regionlist` allows to define a new volume constraint in the region. The label in the `regionlist` will be the previous label of region. This parameter and `nbofregions` can't be used with parameter `sizevolume`.

Example:

Example 5.19 (refinesphere.edp)

// file refinesphere.edp

```
load "msh3"
load "tetgen"
load "medit"

mesh Th=square(10,20,[x*pi-pi/2,2*y*pi]); // ]-pi/2,frac-pi2[x]0,2pi[
// a parametrization of a sphere

func f1 =cos(x)*cos(y);
func f2 =cos(x)*sin(y);
func f3 = sin(x);

// partiel derivative of the parametrization DF

func f1x=sin(x)*cos(y);
func f1y=-cos(x)*sin(y);
func f2x=-sin(x)*sin(y);
func f2y=cos(x)*cos(y);
func f3x=cos(x);
func f3y=0;

// M = DF^t DF

func m11=f1x^2+f2x^2+f3x^2;
func m21=f1x*f1y+f2x*f2y+f3x*f3y;
func m22=f1y^2+f2y^2+f3y^2;

func perio=[[4,y],[2,y],[1,x],[3,x]];
real hh=0.1;
real vv= 1/square(hh);
verbosity=2;
Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
plot(Th,wait=1);

verbosity=2;
```

```

//      construction of the surface of spheres

real Rmin  = 1.;
func flmin = Rmin*f1;
func f2min = Rmin*f2;
func f3min = Rmin*f3;

mesh3 Th3=movemesh23(Th,transfo=[flmin,f2min,f3min]);

real[int] domain = [0.,0.,0.,145,0.01];
mesh3 Th3sph=tetg(Th3,switch="paAAQYY",nbofregions=1,regionlist=domain);

int[int] newlabel = [145,18];
real[int] domainrefine = [0.,0.,0.,145,0.0001];
mesh3 Th3sphrefine=tetgreconstruction(Th3sph,switch="raAQ",reftet=newlabel,
nbofregions=1,regionlist=domain,refinesizeofvolume=0.0001);

int[int] newlabel2 = [145,53];
func fsize = 0.01/(( 1 + 5*sqrt( (x-0.5)^2+(y-0.5)^2+(z-0.5)^2 ) ^3);
mesh3 Th3sphrefine2=tetgreconstruction(Th3sph,switch="raAQ",reftet=newlabel2,
sizeofvolume=fsize);

medit ('`sphere``',Th3sph);
medit ('`isotroperefine``',Th3sphrefine);
medit ('`anisotroperefine``',Th3sphrefine2);

```

5.10.4 Moving mesh in three dimensions

Meshes in three dimensions can be translated rotated and deformed using the command line **movemesh** as in the 2D case (see section **movemesh** in chapter 5). If Ω is tetrahedrized as $T_h(\Omega)$, and $\Phi(x, y) = (\Phi_1(x, y, z), \Phi_2(x, y, z), \Phi_3(x, y, z))$ is a displacement vector then $\Phi(T_h)$ is obtained by

```
mesh3 Th = movemesh( Th, [Φ1, Φ2, Φ3], ... );
```

The parameters of **movemesh** in three dimensions are

region = set integer label of tetrahedra. 0 by default.

label = set the label of faces of border. This parameters is initialized as label for the keyword change (5.3).

facemerge = An integer expression. When you transform a mesh, some faces can be merged. This parameters equals to one if merge's faces is considered. Otherwise equals to zero. By default, this parameter is equals to 1.

ptmerge = A real expression. When you transform a mesh, some points can be merged. This parameters is the criteria to define two merging points. By default, we use

$$ptmerge = 1e - 7 \text{ Vol}(B),$$

where B is the smallest axis parallel boxes containing the discretion domain of Ω and $\text{Vol}(B)$ is the volume of this box.

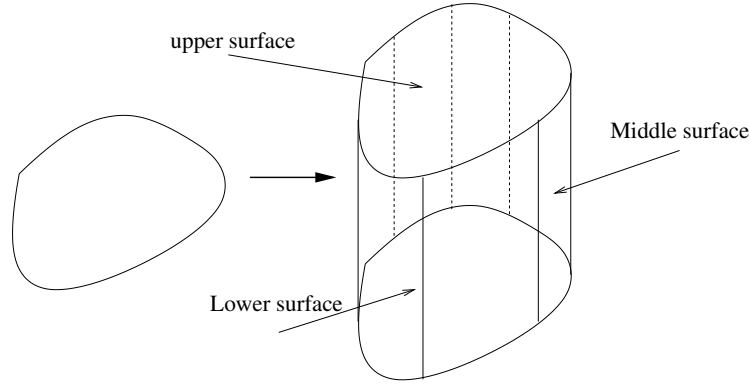


Figure 5.34: Example of Layer mesh in three dimension.

orientation= An integer expression (1 by default) , to reverse or not the orientation of tet if not positive.

An example of this command can be found in the file "Poisson3d.edp" located in the directory examples++-3d.

5.10.5 Layer mesh

In this section, we present the command line to obtain a Layer mesh: `buildlayermesh`. This mesh is obtained by extending a two dimensional mesh in the z -axis.

The domain Ω_{3d} defined by the layer mesh is equal to $\Omega_{3d} = \Omega_{2d} \times [zmin, zmax]$ where Ω_{2d} is the domain define by the two dimensional mesh, $zmin$ and $zmax$ are function of Ω_{2d} in R that defines respectively the lower surface and upper surface of Ω_{3d} .

For a vertex of a two dimensional mesh $V_i^{2d} = (x_i, y_i)$, we introduce the number of associated vertices in the z -axis $M_i + 1$. We denote by M the maximum of M_i over the vertices of the two dimensional mesh. This value are called the number of layers (if $\forall i, M_i = M$ then there are M layers in the mesh of Ω_{3d}). V_i^{2d} generated $M + 1$ vertices which are defined by

$$\forall j = 0, \dots, M, \quad V_{i,j}^{3d} = (x_i, y_i, \theta_i(z_{i,j})),$$

where $(z_{i,j})_{j=0,\dots,M}$ are the $M + 1$ equidistant points on the interval $[zmin(V_i^{2d}), zmax(V_i^{2d})]$:

$$z_{i,j} = j \delta\alpha + zmin(V_i^{2d}), \quad \delta\alpha = \frac{zmax(V_i^{2d}) - zmin(V_i^{2d})}{M}.$$

The function θ_i , defined on $[zmin(V_i^{2d}), zmax(V_i^{2d})]$, is given by

$$\theta_i(z) = \begin{cases} \theta_{i,0} & \text{if } z = zmin(V_i^{2d}), \\ \theta_{i,j} & \text{if } z \in [\theta_{i,j-1}, \theta_{i,j}], \end{cases}$$

with $(\theta_{i,j})_{j=0,\dots,M_i}$ are the $M_i + 1$ equidistant points on the interval $[zmin(V_i^{2d}), zmax(V_i^{2d})]$.

Set a triangle $K = (V_{i1}^{2d}, V_{i2}^{2d}, V_{i3}^{2d})$ of the two dimensional mesh. K is associated with a triangle on the upper surface (resp. on the lower surface) of layer mesh: $(V_{i1,M}^{3d}, V_{i2,M}^{3d}, V_{i3,M}^{3d})$ (resp. $(V_{i1,0}^{3d}, V_{i2,0}^{3d}, V_{i3,0}^{3d})$).

Also K is associated with M volume prismatic elements which are defined by

$$\forall j = 0, \dots, M, \quad H_j = (V_{i1,j}^{3d}, V_{i2,j}^{3d}, V_{i3,j}^{3d}, V_{i1,j+1}^{3d}, V_{i2,j+1}^{3d}, V_{i3,j+1}^{3d}).$$

Theses volume elements can have some merged point:

- 0 merged point : prism
- 1 merged points : pyramid
- 2 merged points : tetrahedra
- 3 merged points : no elements

The elements with merged points are called degenerate elements. To obtain a mesh with tetrahedra, we decompose the pyramid into two tetrahedra and the prism into three tetrahedra. These tetrahedra are obtained by cutting the quadrilateral face of pyramid and prism with the diagonal which have the vertex with the maximum index (see [8] for the reason of this choice).

The triangles on the middle surface obtained with the decomposition of the volume prismatic elements are the triangles generated by the edges on the border of the two dimensional mesh. The label of triangles on the border elements and tetrahedra are defined with the label of these associated elements.

The arguments of `buildlayermesh` is a two dimensional mesh and the number of layers M . The parameters of this command are:

zbound = `[zmin,zmax]` where `zmin` and `zmax` are functions expression. These functions define the lower surface mesh and upper mesh of surface mesh.

coef = A function expression between `[0,1]`. This parameter is used to introduce degenerate element in mesh. The number of associated points or vertex V_i^{2d} is the integer part of $coef(V_i^{2d})M$.

region = This vector is used to initialize the region of tetrahedra. This vector contains successive pair of the 2d region number at index $2i$ and the corresponding 3d region number at index $2i + 1$, like (5.3). become the

labelmid = This vector is used to initialize the 3d labels number of the vertical face or mid face from the 2d label number. This vector contains successive pair of the 2d label number at index $2i$ and the corresponding 3d label number at index $2i + 1$, like (5.3).

labelup = This vector is used to initialize the 3d label numbers of the upper/top face from the 2d region number. This vector contains successive pair of the 2d region number at index $2i$ and the corresponding 3d label number at index $2i + 1$, like (5.3).

labeldown = Same as the previous case but for the lower/down face label .

Moreover, we also add post processing parameters that allow to moving the mesh. These parameters correspond to parameters `transfo`, `facemerge` and `ptmerge` of the command line `movemesh`. The vector `region`, `labelmid`, `labelup` and `labeldown` These vectors are composed of n_l successive pairs of number O_i, N_l where n_l is the number (label or region) that we want to get. An example of this command line is given in `buildlayermesh.edp`.

Example 5.20 (cube.idp)

```
load "medit"
load "msh3"
func mesh3 Cube(int[int] & NN,real[int,int] &BB ,int[int,int] & L)
{
```

```

//      first build the 6 faces of the hex.
real x0=BB(0,0),x1=BB(0,1);
real y0=BB(1,0),y1=BB(1,1);
real z0=BB(2,0),z1=BB(2,1);

int nx=NN[0],ny=NN[1],nz=NN[2];
mesh Thx = square(nx,ny,[x0+(x1-x0)*x,y0+(y1-y0)*y]);

int[int] rup=[0,L(2,1)], rdown=[0,L(2,0)],
      rmid=[1,L(1,0), 2,L(0,1), 3, L(1,1), 4, L(0,0) ];
mesh3 Th=buildlayers(Thx,nz, zbound=[z0,z1],
      labelmid=rmid, labelup = rup,
      labeldown = rdown);

return Th;
}

```

The unit cube example:

```

include "Cube.idp"
int[int] NN=[10,10,10]; //      the number of step in each direction
real [int,int] BB=[ [0,1], [0,1], [0,1] ]; //      bounding box
int [int,int] L=[ [1,2], [3,4], [5,6] ]; //      the label of the 6 face
//      left, right,
//      front, back, down, right
mesh3 Th=Cube(NN,BB,L);
medit ("Th",Th); //      see figure 5.35

```

The cone example (an axisymtric mesh on a triangle with degenerateness).

Example 5.21 (cone.edp)

```

load "msh3"
load "medit"

//      cone using buildlayers with a triangle
real RR=1,HH=1;
border Taxe(t=0,HH){x=t;y=0;label=0;};
border Hypo(t=1,0){x=HH*t;y=RR*t;label=1;};
border Vert(t=0,RR){x=HH;y=t;label=2;};
int nn=10; real h= 1./nn;
mesh Th2=buildmesh( Taxe(HH*nn)+ Hypo(sqrt(HH*HH+RR*RR)*nn) + Vert(RR*nn) );
plot (Th2,wait=1); //      the 2d mesh

int MaxLayersT=(int(2*pi*RR/h)/4)*4; //      number of layers
real zminT = 0, zmaxT = 2*pi; //      height 2*pi
func fx= y*cos(z); func fy= y*sin(z); func fz= x;
int[int] r1T=[0,0], r2T=[0,0,2,2], r4T=[0,2];

//      trick function:
func deg= max(.01,y/max(x/HH,0.4) /RR); //      the function defined the
//      proportion
//      of number layer close to axis with reference MaxLayersT
mesh3 Th3T=buildlayers(Th2,coef= deg, MaxLayersT,
      zbound=[zminT,zmaxT],transfo=[fx,fy,fz],
      facemerge=0, region=r1T, labelmid=r2T);
medit ("cone",Th3T); //      see figure 5.36

```

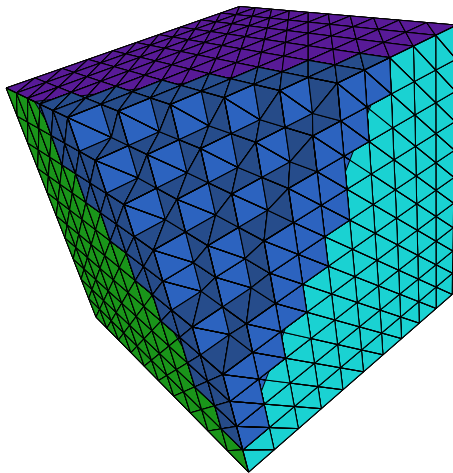


Figure 5.35: the mesh of a cube made with cube.edp

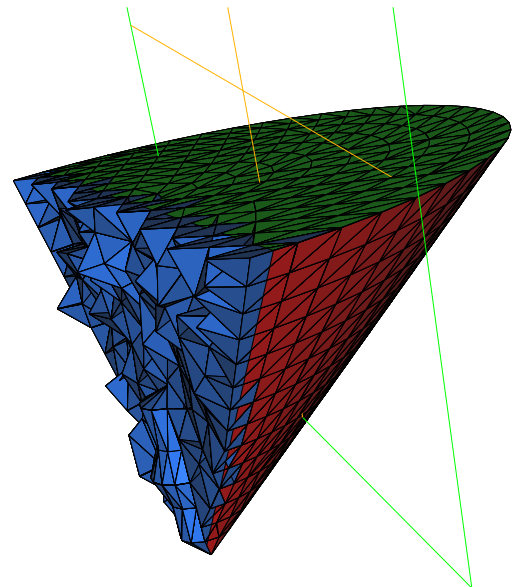


Figure 5.36: the mesh of a cone made with cone.edp

Example 5.22 (buildlayermesh.edp)

```

// file buildlayermesh.edp

load "msh3"
load "tetgen"

// Test 1

int C1=99, C2=98;
border C01(t=0,pi){ x=t; y=0; label=1;}
border C02(t=0,2*pi){ x=pi; y=t; label=1;}
border C03(t=0,pi){ x=pi-t; y=2*pi; label=1;}
border C04(t=0,2*pi){ x=0; y=2*pi-t; label=1;}

border C11(t=0,0.7){ x=0.5+t; y=2.5; label=C1;}
border C12(t=0,2){ x=1.2; y=2.5+t; label=C1;}
border C13(t=0,0.7){ x=1.2-t; y=4.5; label=C1;}
border C14(t=0,2){ x=0.5; y=4.5-t; label=C1;}

border C21(t=0,0.7){ x= 2.3+t; y=2.5; label=C2;}
border C22(t=0,2){ x=3; y=2.5+t; label=C2;}
border C23(t=0,0.7){ x=3-t; y=4.5; label=C2;}
border C24(t=0,2){ x=2.3; y=4.5-t; label=C2;}

mesh Th=buildmesh( C01(10)+C02(10)+ C03(10)+C04(10)
+ C11(5)+C12(5)+C13(5)+C14(5)

```



```

+ C21(-5)+C22(-5)+C23(-5)+C24(-5));

mesh Ths=buildmesh(    C01(10)+C02(10)+ C03(10)+C04(10)
+ C11(5)+C12(5)+C13(5)+C14(5) );

                                //    construction of a box with one hole and two regions
func zmin=0.;
func zmax=1.;
int MaxLayer=10;

func XX = x*cos(y);
func YY = x*sin(y);
func ZZ = z;

int[int] r1=[0,41], r2=[98,98, 99,99, 1,56];
int[int] r3=[4,12];                                //    The triangles of uppper surface mesh
//    generated by the triangle in the 2D region of mesh Th of label 4 as
//    label 12.
int[int] r4=[4,45];                                //    The triangles of lower surface mesh
//    generated by the triangle in the 2D region of mesh Th of label 4 as
//    label 45.

mesh3 Th3=buildlayers( Th, MaxLayer, zbound=[zmin,zmax], region=r1,
                        labelmid=r2, labelup = r3, labeldown = r4 );
savemesh(Th3,"box2region1hole.mesh");
                                //    construction of a sphere with TetGen
func XX1 = cos(y)*sin(x);
func YY1 = sin(y)*sin(x);
func ZZ1 = cos(x);
string test="paACQ";
cout << "test=" << test << endl;
mesh3 Th3sph=tetgtransfo(Ths,transfo=[XX1,YY1,ZZ1],switch=test,nbofregions=1,
                        regionlist=domain);
savemesh(Th3sph,"sphere2region.mesh");

```

5.11 Meshing examples

Example 5.23 (lac.edp) // file "lac.edp"

```

load ``msh3``
int nn=5;
border cc(t=0,2*pi){x=cos(t);y=sin(t);label=1;}
mesh Th2 = buildmesh(cc(100));
fespace Vh2(Th2,P2);
Vh2 ux,uy,p2;
int[int] rup=[0,2], rdlow=[0,1], rmid=[1,1,2,1,3,1,4,1];
func zmin = 2-sqrt(4-(x*x+y*y));
func zmax = 2-sqrt(3.);

mesh3 Th = buildlayers(Th2,nn,
                        coeff = max((zmax-zmin)/zmax, 1./nn),
                        zbound=[zmin,zmax],

```

```

    labelmid=rmid;
    labelup=rup;
    labeldown=rld;
savemesh(Th, 'Th.meshb');
exec('`medit Th; Th.meshb`');

```

Example 5.24 (tetgenholeregion.edp)

```

// file `tetgenholeregion.edp`
load "msh3"
load "tetgen"

mesh Th=square(10,20,[x*pi-pi/2,2*y*pi]); // ] $-\frac{\pi}{2}, \frac{\pi}{2}$ [x]0,2 $\pi$ [
// a parametrization of a sphere

func f1 =cos(x)*cos(y);
func f2 =cos(x)*sin(y);
func f3 = sin(x);

// partiel derivative of the parametrization DF
func f1x=sin(x)*cos(y);
func f1y=-cos(x)*sin(y);
func f2x=-sin(x)*sin(y);
func f2y=cos(x)*cos(y);
func f3x=cos(x);
func f3y=0;

// M = DFtDF
func m11=f1x^2+f2x^2+f3x^2;
func m21=f1x*f1y+f2x*f2y+f3x*f3y;
func m22=f1y^2+f2y^2+f3y^2;

func perio=[[4,y],[2,y],[1,x],[3,x]];
real hh=0.1;
real vv= 1/square(hh);
verbosity=2;
Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
plot(Th,wait=1);

verbosity=2;

// construction of the surface of spheres
real Rmin = 1.;
func f1min = Rmin*f1;
func f2min = Rmin*f2;
func f3min = Rmin*f3;

mesh3 Th3sph = movemesh23(Th,transfo=[f1min,f2min,f3min]);

real Rmax = 2.;
func f1max = Rmax*f1;
func f2max = Rmax*f2;
func f3max = Rmax*f3;

mesh3 Th3sph2 = movemesh23(Th,transfo=[f1max,f2max,f3max]);

cout << "addition" << endl;
mesh3 Th3 = Th3sph+Th3sph2;

```

```

real[int] domain2 = [1.5,0.,0.,145,0.001,0.5,0.,0.,18,0.001];
cout << "======" << endl;
cout << " tetgen call without hole " << endl;
cout << "======" << endl;
mesh3 Th3fin = tetg(Th3,switch="paAAQYY",nbofregions=2,regionlist=domain2);
cout << "======" << endl;
cout << "finish tetgen call without hole" << endl;
cout << "======" << endl;
savemesh(Th3fin,"spherewithtworegion.mesh");

real[int] hole = [0.,0.,0.];
real[int] domain = [1.5,0.,0.,53,0.001];
cout << "======" << endl;
cout << " tetgen call with hole " << endl;
cout << "======" << endl;
mesh3 Th3finhole=tetg(Th3,switch="paAAQYY",nbofholes=1,holelist=hole,
nbofregions=1,regionlist=domain);
cout << "======" << endl;
cout << "finish tetgen call with hole " << endl;
cout << "======" << endl;
savemesh(Th3finhole,"spherewithahole.mesh");

```

5.11.1 Build a 3d mesh of a cube with a balloon

First the MeshSurface.idp file to build boundary mesh of a Hexaedra and of a Sphere.

```

func mesh3 SurfaceHex(int[int] & N,real[int,int] & B ,int[int,int] & L,int orientation)
{
    real x0=B(0,0),x1=B(0,1);
    real y0=B(1,0),y1=B(1,1);
    real z0=B(2,0),z1=B(2,1);

    int nx=N[0],ny=N[1],nz=N[2];

    mesh Thx = square(ny,nz,[y0+(y1-y0)*x,z0+(z1-z0)*y]);
    mesh Thy = square(nx,nz,[x0+(x1-x0)*x,z0+(z1-z0)*y]);
    mesh Thz = square(nx,ny,[x0+(x1-x0)*x,y0+(y1-y0)*y]);

    int[int] refx=[0,L(0,0)],refX=[0,L(0,1)]; // Xmin, Ymax faces labels
    renumbering
    int[int] refy=[0,L(1,0)],refY=[0,L(1,1)]; // Ymin, Ymax faces labesl
    renumbering
    int[int] refz=[0,L(2,0)],refZ=[0,L(2,1)]; // Zmin, Zmax faces labels
    renumbering

    mesh3 Thx0 = movemesh23(Thx,transfo=[x0,x,y],orientation=-orientation,label=refx);
    mesh3 Thx1 = movemesh23(Thx,transfo=[x1,x,y],orientation=+orientation,label=refX);
    mesh3 Thy0 = movemesh23(Thy,transfo=[x,y0,y],orientation=+orientation,label=refy);
    mesh3 Thy1 = movemesh23(Thy,transfo=[x,y1,y],orientation=-orientation,label=refY);
    mesh3 Thz0 = movemesh23(Thz,transfo=[x,y,z0],orientation=-orientation,label=refz);
    mesh3 Thz1 = movemesh23(Thz,transfo=[x,y,z1],orientation=+orientation,label=refZ);
    mesh3 Th= Thx0+Thx1+Thy0+Thy1+Thz0+Thz1;
    return Th;
}

```

```

}

func mesh3 Sphere(real R,real h,int L,int orientation)
{
    mesh Th=square(10,20,[x*pi-pi/2,2*y*pi]); //  $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi]$ 
                                                // a parametrization of a sphere

    func f1 =cos(x)*cos(y);
    func f2 =cos(x)*sin(y);
    func f3 = sin(x);

                                                // partiel derivative

    func f1x=sin(x)*cos(y);
    func f1y=-cos(x)*sin(y);
    func f2x=-sin(x)*sin(y);
    func f2y=cos(x)*cos(y);
    func f3x=cos(x);
    func f3y=0;

                                                // the metric on the sphere  $M = DF^t DF$ 

    func m11=f1x^2+f2x^2+f3x^2;
    func m21=f1x*f1y+f2x*f2y+f3x*f3y;
    func m22=f1y^2+f2y^2+f3y^2;

    func perio=[[4,y],[2,y],[1,x],[3,x]]; // to store the periodic condition

    real hh=h/R; // hh mesh size on unite sphere
    real vv= 1/square(hh);
    Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
    Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
    Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
    Th=adaptmesh(Th,m11*vv,m21*vv,m22*vv,IsMetric=1,periodic=perio);
    int[int] ref=[0,L];

    mesh3 ThS= movemesh23(Th,transfo=[f1*R,f2*R,f3*R],orientation=orientation,refface=ref);
    return ThS;
}

```

The test of the two functions and the call to tetgen mesh generator

```

load "tetgen"
include "MeshSurface.idp"
    real hs = 0.1; // mesh size on sphere
    int[int] N=[20,20,20];
    real [int,int] B=[[-1,1],[-1,1],[-1,1]];
    int [int,int] L=[[1,2],[3,4],[5,6]];
    mesh3 ThH = SurfaceHex(N,B,L,1);
    mesh3 ThS =Sphere(0.5,hs,7,1); // "gluing" surface meshes to tolat
    boundary meshes

    mesh3 ThHS=ThH+ThS;
    savemesh(ThHS,"Hex-Sphere.mesh");
    exec("ffmedit Hex-Sphere.mesh;rm Hex-Sphere.mesh"); // see 5.37

    real voltet=(hs^3)/6.;
    cout << " voltet = " << voltet << endl;
    real[int] domaine = [0,0,0,1,voltet,0,0,0.7,2,voltet];

```

```

mesh3 Th = tetg(ThHS,switch="pqaAAYYQ",nbofregions=2,regionlist=domaine);
medit ("Cube-With-Ball",Th); // see 5.38

```

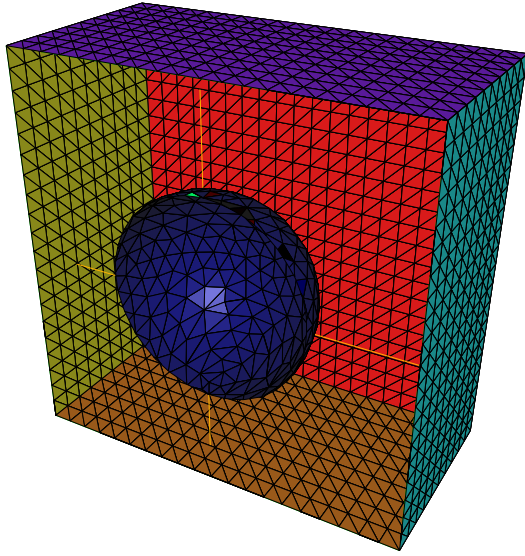


Figure 5.37: The surface mesh of the Hex with internal Sphere

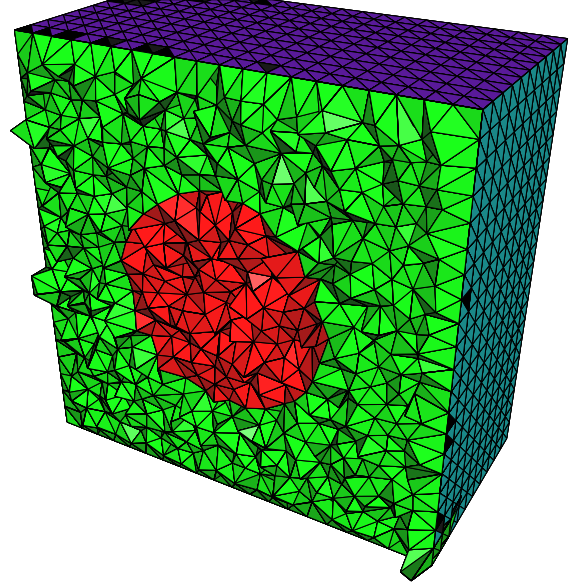


Figure 5.38: The tet mesh of the cube with internal ball

5.12 The output solution formats .sol and .solb

With the keyword `savesol`, we can store a scalar functions, a scalar FE functions, a vector fields, a vector FE fields, a symmetric tensor and a symmetric FE tensor.. Such format is used in `medit`.

extension file .sol The first two lines of the file are

- `MeshVersionFormatted 0`
- `Dimension (I) dim`

The following fields begin with one of the following keyword: `SolAtVertices`, `SolAtEdges`, `SolAtTriangles`, `SolAtQuadrilaterals`, `SolAtTetrahedra`, `SolAtPentahedra`, `SolAtHexahedra`.

In each field, we give then in the next line the number of elements in the solutions (`SolAtVertices`: number of vertices, `SolAtTriangles`: number of triangles, ...). In other lines, we give the number of solutions, the type of solution (1: scalar, 2: vector, 3: symmetric tensor). And finally, we give the values of the solutions on the elements.

The file must be ended with the keyword `End`.

The real element of symmetric tensor

$$ST^{3d} = \begin{pmatrix} ST_{xx}^{3d} & ST_{xy}^{3d} & ST_{xz}^{3d} \\ ST_{yx}^{3d} & ST_{yy}^{3d} & ST_{yz}^{3d} \\ ST_{zx}^{3d} & ST_{zy}^{3d} & ST_{zz}^{3d} \end{pmatrix} \quad ST^{2d} = \begin{pmatrix} ST_{xx}^{2d} & ST_{xy}^{2d} \\ ST_{yx}^{2d} & ST_{yy}^{2d} \end{pmatrix} \quad (5.5)$$

stored in the extension .sol are respectively $ST_{xx}^{3d}, ST_{yx}^{3d}, ST_{yy}^{3d}, ST_{zx}^{3d}, ST_{zy}^{3d}, ST_{zz}^{3d}$ and $ST_{xx}^{2d}, ST_{yx}^{2d}, ST_{yy}^{2d}$. An example of field with the keyword SolAtTetrahedra:

- SolAtTetrahedra
 - (I) NbOfTetrahedrons
 - nbsol typesol¹ ... typesolⁿ
 - $\left(\left(\left(U_{ij}^k, \quad \forall i \in \{1, \dots, \text{nbrealsol}^k\} \right), \quad \forall k \in \{1, \dots, \text{nbsol}\} \right) \quad \forall j \in \{1, \dots, \text{NbOfTetrahedrons}\} \right)$

where

- nbsol is an integer equal to the number of solutions
- typesol^k, type of the solution number k , is
 - typesol^k = 1 the solution k is scalar.
 - typesol^k = 2 the solution k is vectorial.
 - typesol^k = 3 the solution k is a symmetric tensor or symmetric matrix.
- nbrealsol^k number of real to discribe solution number k is
 - nbrealsol^k = 1 the solution k is scalar.
 - nbrealsol^k = dim the solution k is vectorial (dim is the dimension of the solution).
 - nbrealsol^k = $\text{dim} * (\text{dim} + 1) / 2$ the solution k is a symmetric tensor or symmetric matrix.
- U_{ij}^k is a real equal to the value of the component i of the solution k at tetrahedra j on the associated mesh.

This field is written with the notation of Section 12.1. The format .solb is the same as format .sol but in binary (read/write is faster, storage is less).

A real scalar functions $f1$, a vector fields $\Phi = [\Phi1, \Phi2, \Phi3]$ and a symmetric tensor ST^{3d} (5.5) at the vertices of the three dimensional mesh Th3 is stored in the file "f1PhiTh3.sol" using

```
savesol("f1PhiST3dTh3.sol", Th3, f1, [\Phi1, \Phi2, \Phi3], VV3, order=1);
```

where $VV3 = [ST_{xx}^{3d}, ST_{yx}^{3d}, ST_{yy}^{3d}, ST_{zx}^{3d}, ST_{zy}^{3d}, ST_{zz}^{3d}]$. For a two dimensional mesh Th, A real scalar functions $f2$, a vector fields $\Psi = [\Psi1, \Psi2]$ and a symmetric tensor ST^{2d} (5.5) at triangles is stored in the file "f2PsiST2dTh3.solb" using

```
savesol("f2PsiST2dTh3.solb", Th, f2, [\Psi1, \Psi2], VV2, order=0);
```

where $VV2 = [ST_{xx}^{2d}, ST_{yx}^{2d}, ST_{yy}^{2d}]$. The arguments of savesol functions are the name of a file, a mesh and solutions. These arguments must be given in this order.

The parmameters of this keyword are

order = 0 is the solution is given at the center of gravity of elements. 1 is the solution is given at the vertices of elements.

In the file, solutions are stored in this order : scalar solutions, vector solutions and finally symmetric tensor solutions.

5.13 medit

The keyword `medit` allows to display a mesh alone or a mesh and one or several functions defined on the mesh using the Pascal Frey's freeware `medit`. `Medit` opens its own window and uses OpenGL extensively. Naturally to use this command `medit` must be installed.

A visualization with `medit` of scalar solutions $f1$ and $f2$ continuous, piecewise linear and known at the vertices of the mesh Th is obtained using

```
medit("sol1 sol2",Th, f1, f2, order=1);
```

The first plot named "sol1" display $f1$. The second plot names "sol2" display $f2$.

The arguments of function `medit` are the name of the different scenes (separated by a space) of `medit`, a mesh and solutions. Each solution is associated with one scene. The scalar, vector and symmetric tensor solutions are specified in the format described in the section dealing with the keyword `savesol`.

The parameters of this command line are

order = 0 is the solution is given at the center of gravity of elements. 1 is the solution is given at the vertices of elements.

meditff = set the name of execute command of `medit`. By default, this string is `medit`.

save = set the name of a file `.sol` or `.solb` to save solutions.

This command line allows also to represent two different meshes and solutions on them in the same windows. The nature of solutions must be the same. Hence, we can visualize in the same window the different domains in a domain decomposition method for instance. A visualization with `medit` of scalar solutions $h1$ and $h2$ at vertices of the mesh $Th1$ and $Th2$ respectively are obtained using

```
medit("sol2domain",Th1, h1, Th2, h2, order=1);
```

Example 5.25 (meditddm.edp)

```
load "medit" // meditddm.edp

// Initial Problem:
// Resolution of the following EDP:
//  $-\Delta u_s = f$  on  $\Omega = \{(x,y) | 1 \leq \sqrt{x^2 + y^2} \leq 2\}$ 
//  $-\Delta u_1 = f1$  on  $\Omega_1 = \{(x,y) | 0.5 \leq \sqrt{x^2 + y^2} \leq 1\}$ 
//  $u = 1$  on  $\Gamma$  + Null Neumann condition on  $\Gamma_1$  and on  $\Gamma_2$ 
// We find the solution  $u$  in solving two EDP defined on domain  $\Omega$  and  $\Omega_1$ 
// This solution is visualize with medit

verbosity=3;

border Gamma(t=0,2*pi){x=cos(t); y=sin(t); label=1;};
border Gamma1(t=0,2*pi){x=2*cos(t); y=2*sin(t); label=2;};
border Gamma2(t=0,2*pi){x=0.5*cos(t); y=0.5*sin(t); label=3;};

// construction of mesh of domain  $\Omega$ 
mesh Th=buildmesh(Gamma1(40)+Gamma(-40));

fespace Vh(Th,P2);
```

```

func f=sqrt(x*x+y*y);
Vh us,v;
macro Grad2(us) [dx(us),dy(us)] // EOM

problem Lap2dOmega(us,v,init=false)=int2d(Th) (Grad2(v)' *Grad2(us))
- int2d(Th) (f*v)+on(1,us=1) ;

// Definition of EDP defined on the domain  $\Omega$ 
//  $-\Delta u_s = f_1$  on  $\Omega_1$ ,  $u_s = 1$  on  $\Gamma_1$ ,  $\frac{\partial u_s}{\partial n} = 0$  on  $\Gamma_2$ 
Lap2dOmega;

// construction of mesh of domain  $\Omega_1$ 
mesh Th1=buildmesh(Gamma(40)+Gamma2(-40));

fespace Vh1(Th1,P2);
func f1=10*sqrt(x*x+y*y);
Vh1 u1,v1;
macro Grad21(u1) [dx(u1),dy(u1)] // EOM

problem Lap2dOmegal(u1,v1,init=false)=int2d(Th1) (Grad21(v1)' *Grad21(u1))
- int2d(Th1) (f1*v1)+on(1,u1=1) ;

// Resolution of EDP defined on the domain  $\Omega_1$ 
//  $-\Delta u_1 = f_1$  on  $\Omega_1$ ,  $u-1 = 1$  on  $\Gamma_1$ ,  $\frac{\partial u_1}{\partial n} = 0$  on  $\Gamma_2$ 
Lap2dOmegal;

// visualisation of solution of the initial problem
medit("solution",Th,us,Th1,u1,order=1,save="testsavemedit.solb");

```

Example 5.26 (StockesUzawa.edp) // signe of pressure is correct

```

assert(version>1.18);
real s0=clock();
mesh Th=square(10,10);
fespace Xh(Th,P2),Mh(Th,P1);
Xh u1,u2,v1,v2;
Mh p,q,ppp;

varf bx(u1,q) = int2d(Th) ( (dx(u1)*q) );
varf by(u1,q) = int2d(Th) ( (dy(u1)*q) );
varf a(u1,u2)= int2d(Th) ( dx(u1)*dx(u2) + dy(u1)*dy(u2) )
+ on(1,2,4,u1=0) + on(3,u1=1) ;

Xh bcl; bcl[] = a(0,Xh);
Xh b;

matrix A= a(Xh,Xh,solver=CG);
matrix Bx= bx(Xh,Mh);
matrix By= by(Xh,Mh);
Xh bcx=1,bcy=0;

func real[int] divup(real[int] & pp)
{
  int verb=verbosity;
  verbosity=0;

```



```

    b[] = Bx'*pp; b[] += bc1[] .*bcx[];
    u1[] = A^-1*b[];
    b[] = By'*pp; b[] += bc1[] .*bcy[];
    u2[] = A^-1*b[];
    ppp[] = Bx*u1[];
    ppp[] += By*u2[];
    verbosity=verb;
    return ppp[] ;
};
p=0;q=0;u1=0;v1=0;

LinearCG(divup,p[],q[],eps=1.e-6,nbiter=50);

divup(p[]);

plot([u1,u2],p,wait=1,value=true,coef=0.1);
medit("velocity pressure",Th,[u1,u2],p,order=1);

```

5.14 Mshmet

Mshmet is a software developed by P. Frey that allows to compute an anisotropic metric based on solutions (i.e. Hessian-based). This software can return also an isotropic metric. Moreover, mshmet can construct also a metric suitable for level sets interface capturing. The solution can be defined on 2D or 3D structured/unstructured meshes. For example, the solution can be an error estimate of a FE solutions.

Solutions for mshmet are given as an argument. The solution can be a func, a vector func, a symmetric tensor, a FE func, a FE vector func and a FE symmetric tensor. The symmetric tensor argument is defined as this type of data for datasol argument. This software accepts more than one solution.

For example, the metric M computed with mshmet for the solution u defined on the mesh Th is obtained by writing.

```

fespace Vh(Th,P1);
Vh u;                                     // a scalar FE func
real[int] M = mshmet(Th,u);

```

The parameters of the keyword mshmet are :

- `normalization` = do a normalisation of all solution in $[0, 1]$.
- `aniso` = build aniso metric if 1 (default 0: iso)
- `levelset` = build metric for level set method (default: false)
- `verbosity` = <l>
- `nbregul` = <l> number of regularization's iteration of solutions given (default 0).
- `hmin` = <d>
- `hmax` = <d>
- `err` = <d> level of error.
- `width` = <d> the width

- `metric` = a vector of double. This vector contains an initial metric given to `mshmet`. The structure of the metric vector is described in the next paragraph.
- `loptions` = a vector of integer of size 7. This vector contains the integer parameters of `mshmet` (for expert only).
 - `loptions(0)`: normalization (default 1).
 - `loptions(1)`: isotropic parameters (default 0). 1 for isotropic metric results otherwise 0.
 - `loptions(2)`: level set parameters (default 0). 1 for building level set metric otherwise 0.
 - `loptions(3)`: debug parameters (default 0). 1 for turning on debug mode otherwise 0.
 - `loptions(4)`: level of verbosity (default 10).
 - `loptions(5)`: number of regularization's iteration of solutions given (default 0).
 - `loptions(6)`: previously metric parameter (default 0). 1 for using previous metric otherwise 0.
- `doptions` = a vector of double of size 4. This vector contains the real parameters of `mshmet` (for expert only).
 - `doptions(0)`: `hmin` : min size parameters (default 0.01).
 - `doptions(1)`: `hmax` : max size parameters (default 1.0).
 - `doptions(2)`: `eps` : tolerance parameters (default 0.01).
 - `doptions(2)`: `width` : relative width for Level Set ($0 < w < 1$) (default 0.05).

The result of the keyword `mshmet` is a `real[int]` which contains the metric computed by `mshmet` at the different vertices V_i of the mesh.

With nv is the number of vertices, the structure of this vector is

$$M_{iso} = (m(V_0), m(V_1), \dots, m(V_{nv}))^t$$

for a isotropic metric m . For a symmetric tensor metric $h = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$, the parameters `metric` is

$$M_{aniso} = (H(V_0), \dots, H(V_{nv}))^t$$

where $H(V_i)$ is the vector of size 6 defined by `[m11, m21, m22, m31, m32, m33]`

Example 5.27 (`mshmet.edp`)

```
load "mshmet"
load "medit"
load "msh3"

border a(t=0,1.0){x=t; y=0; label=1;};
border b(t=0,0.5){x=1; y=t; label=2;};
border c(t=0,0.5){x=1-t; y=0.5; label=3;};
border d(t=0.5,1){x=0.5; y=t; label=4;};
border e(t=0.5,1){x=1-t; y=1; label=5;};
border f(t=0.0,1){x=0; y=1-t; label=6;};
mesh Th = buildmesh (a(6) + b(4) + c(4) + d(4) + e(4) + f(6));
savemesh(Th, "th.msh");
fespace Vh(Th, P1);
```

```

Vh u,v;
real error=0.01;
problem Problem1(u,v,solver=CG,eps=1.0e-6) =
    int2d(Th,qforder=2) ( u*v*1.0e-10+ dx(u)*dx(v) + dy(u)*dy(v))
    +int2d(Th,qforder=2) ( (x-y)*v);

func zmin=0;
func zmax=1;
int MaxLayer=10;
mesh3 Th3 = buildlayers (Th,MaxLayer,zbound=[zmin,zmax]);
fespace Vh3 (Th3,P2);
fespace Vh3P1 (Th3,P1);
Vh3 u3,v3;
Vh3P1 usol;
problem Problem2(u3,v3,solver=sparse solver) =
    int3d(Th3) ( u3*v3*1.0e-10+ dx(u3)*dx(v3) + dy(u3)*dy(v3) + dz(u3)*dz(v3))
    - int3d(Th3) ( v3) +on(0,1,2,3,4,5,6,u3=0);
Problem2;
cout << u3[].min << " " << u3[].max << endl;
savemesh(Th3,"metrictest.bis.mesh");
savesol("metrictest.sol",Th3,u3);

real[int] bb=mshmet(Th3,u3);
cout << bb << endl;
for(int ii=0; ii<Th3.nv; ii++)
    usol[][ii]=bb[ii];
savesol("metrictest.bis.sol",Th3,usol);

```

5.15 FreeYams

FreeYams is a surface mesh adaptation software which is developed by P. Frey. This software is a new version of yams. The adapted surface mesh is constructed with a geometric metric tensor field. This field is based on the intrinsic properties of the discrete surface. Also this software allows to construct a simplification of a mesh. This decimation is based on the Hausdorff distance between the initial and the current triangulation. Compared to the software yams, FreeYams can be used also to produce anisotropic triangulations adapted to level set simulations. A technical report on FreeYams is not available yet but a documentation on yams exists at <http://www.ann.jussieu.fr/~frey/software.html> [40].

To call FreeYams in FreeFem++ , we used the keyword `freeyams`. The arguments of this function are the initial mesh and/or metric. The metric with `freeyams` are a function, a FE function, a symmetric tensor function, a symmetric tensor FE function or a vector of double. If the metric is vector of double, this data must be given in `metric` parameter. Otherwise, the metric is given in the argument.

For example, the adapted mesh of *Thinit* defined by the metric *u* defined as FE function is obtained in writing.

```

fespace Vh(Thinit,P1);
Vh u;
mesh3 Th=freeyams(Thinit,u);

```

The symmetric tensor argument for `freeyams` keyword is defined as this type of data for `datasol` argument.

- `aniso` = `` aniso or iso metric (default 0, iso)
- `mem` = `<l>` memory of for freeyams in Mb (default -1, freeyams choose)
- `hmin` = `<d>`
- `hmax` = `<d>`
- `gradation` = `<d>`
- `option` = `<l>`
 - 0** : mesh optimization (smoothing+swapping)
 - 1** : decimation+enrichment adapted to a metric map. (default)
 - 1** : decimation adapted to a metric map.
 - 2** : decimation+enrichment with a Hausdorff-like method
 - 2** : decimation with a Hausdorff-like method
 - 4** : split triangles recursively.
 - 9** : No-Shrinkage Vertex Smoothing
- `ridgeangle` = `<d>`
- `absolute` = ``
- `verbosity` = `<i>`
- `metric`= vector expression. This parameters contains the metric at the different vertices on the initial mesh. With nv is the number of vertices, this vector is

$$M_{iso} = (m(V_0), m(V_1), \dots, m(V_{nv}))^t$$

for a scalar metric m . For a symmetric tensor metric $h = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$, the parameters `metric` is

$$M_{aniso} = (H(V_0), \dots, H(V_{nv}))^t$$

where $H(V_i)$ is the vector of size 6 defined by `[m11, m21, m22, m31, m32, m33]`

- `loptions`= a vector of integer of size 13. This vectors contains the integer options of FreeYams. (just for the expert)
 - `loptions(0)`: anisotropic parameter (default 0). If you give an anisotropic metric 1 otherwise 0.
 - `loptions(1)`: Finite Element correction parameter (default 0). 1 for *no* Finite Element correction otherwise 0.
 - `loptions(2)`: Split multiple connected points parameter (default 1). 1 for splitting multiple connected points otherwise 0.
 - `loptions(3)`: maximum value of memory size in Mbytes (default -1: the size is given by freeyams).
 - `loptions(4)`: set the value of the connected component which we want to obtain. (Remark: freeyams give an automatic value at each connected component).
 - `loptions(5)`: level of verbosity
 - `loptions(6)`: Create point on straight edge (no mapping) parameter (default 0). 1 for creating point on straight edge otherwise 0.

- loptions(7): validity check during smoothing parameter. This parameter is only used with No-Shrinkage Vertex Smoothing optimization (optimization option parameter 9). 1 for No validity checking during smoothing otherwise 0.
- loptions(8): number of desired's vertices (default -1).
- loptions(9): number of iteration of optimizations (default 30).
- loptions(10): no detection parameter (default 0) . 1 for detecting the ridge on the mesh otherwise 0. The ridge definition is given in the parameter doptions(12).
- loptions(11): no vertex smoothing parameter (default 0). 1 for smoothing the vertices otherwise 0.
- loptions(12): Optimization level parameter (default 0).
 - * 0 : mesh optimization (smoothing+swapping)
 - * 1 : decimation+enrichment adaptated to a metric map.
 - * -1: decimation adaptated to a metric map.
 - * 2 : decimation+enrichment with a Hausdorff-like method
 - * -2: decimation with a Hausdorff-like method
 - * 4 : split triangles recursively.
 - * 9 : No-Shrinkage Vertex Smoothing

doptions= a vector of double of size 11. This vectors contains the real options of freeyams.

- doptions(0): Set the geometric approximation (Tangent plane deviation) (default 0.01).
- doptions(1): Set the lamda parameter (default -1.).
- doptions(2): Set the mu parmeter (default -1.).
- doptions(3): Set the gradation value (Mesh density control) (default 1.3).
- doptions(4): Set the minimal size(hmin) (default -2.0: the size is automatically computed).
- doptions(5): Set the maximal size(hmax) (default -2.0: the size is automatically computed).
- doptions(6): Set the tolerance of the control of Chordal deviation (default -2.0).
- doptions(7): Set the quality of degradation (default 0.599).
- doptions(8): Set the declic parameter (default 2.0).
- doptions(9): Set the angular walton limitation parameter (default 45 degree).
- doptions(10): Set the angular ridge detection (default 45 degree).

Example 5.28 (freeyams.edp)

```
load "msh3"
load "medit"
load "freeyams"
int nn=20;
mesh Th2=square(nn,nn);
fespace Vh2(Th2,P2);
Vh2 ux,uz,p2;
int[int] rup=[0,2], rdown=[0,1], rmid=[1,1,2,1,3,1,4,1];
```

```

real zmin=0, zmax=1;
mesh3 Th=buildlayers(Th2, nn, zbound=[zmin, zmax],
                      reffacemid=rmid, reffaceup = rup, reffacelow = rdown);

mesh3 Th3 = freeyams(Th);
medit("maillagesurfacique", Th3, wait=1);

```

5.16 mmg3d

Mmg3d is a 3D remeshing software developed by C. Dobrzynski and P. Frey (<http://www.math.u-bordeaux1.fr/~dobj/logiciels/mmg3d.php>). To obtain a version of this library send an e-mail at :

cecile.dobrzynski@math.ubordeaux1.fr or pascal.frey@upmc.fr.

This software allows to remesh an initial mesh made of tetrahedra. This initial mesh is adapted to a geometric metric tensor field or to a displacement vector (moving rigid body). The metric can be obtained with mshmet (see section 5.14).

Remark 5 :

- (a) *If no metric is given, an isotropic metric is computed by analyzing the size of the edges in the initial mesh.*
- (b) *if a displacement is given, the vertices of the surface triangles are moved without verifying the geometrical structure of the new surface mesh.*

The parameters of mmg3d are :

- **options**= vector expression. This vector contains the option parameters of mmg3d. It is a vector of 6 values, with the following meaning:
 - (0) optimization parameters : (default 1)
 - 0 : mesh optimization.
 - 1 : adaptation with metric (deletion and insertion vertices) and optimization.
 - 1: adaptation with metric (deletion and insertion vertices) without optimization.
 - 4 : split tetrahedra (be careful modify the surface).
 - 9 : moving mesh with optimization.
 - 9: moving mesh without optimization.
 - (1) debug mode : (default 0)
 - 1 : turn on debug mode.
 - 0 : otherwise.
 - (2) Specify the size of bucket per dimension (default 64)
 - (3) swapping mode : (default 0)
 - 1 : no edge or face flipping.
 - 0 : otherwise.
 - (4) insert points mode : (default 0)
 - 1 : no edge splitting or collapsing and no insert points.
 - 0 : otherwise.

(5) verbosity level (default 3)

- `memory`= integer expression. Set the maximum memory size of new mesh in Mbytes. By default the number of maximum vertices, tetrahedra and triangles are respectively 500 000, 3000 000, 100000 which represent approximately a memory of 100 Mo.
- `metric`= vector expression. This vector contains the metric given at `mmg3d`. It is a vector of size nv or $6\ nv$ respectively for an isotropic and anisotropic metric where nv is the number of vertices in the initial mesh. The structure of `metric` vector is described in the `mshmet`'s section (section 5.14).
- `displacement`= $[\Phi_1, \Phi_2, \Phi_3]$ set the displacement vector of the initial mesh $\Phi(x, y) = [\Phi_1(x, y), \Phi_2(x, y), \Phi_3(x, y)]$.
- `displVect`= sets the vector displacement in a vector expression. This vector contains the displacement at each point of the initial mesh. It is a vector of size $3\ nv$.

An example using this function is given in "mmg3d.edp":

Example 5.29 (mmg3d.edp)

```

//      test mmg3d

load "msh3"
load "medit"
load "mmg3d"
include "../examples++-3d/cube.idp"

int n=6;
int[int] Nxyz=[12,12,12];
real [int,int] Bxyz=[[0.,1.],[0.,1.],[0.,1.]];
int [int,int] Lxyz=[[1,1],[2,2],[2,2]];
mesh3 Th=Cube(Nxyz,Bxyz,Lxyz);

real[int] isometric(Th.nv);{
  for( int ii=0; ii<Th.nv; ii++)
    isometric[ii]=0.17;
}

mesh3 Th3=mmg3d( Th, memory=100, metric=isometric);

medit("init",Th);
medit("isometric",Th3);

```

An example of a moving mesh is given in `fallingspheres.edp`:

Example 5.30 (fallingspheres.edp) `load "msh3"` `load "tetgen"` `load "medit"` `load "mmg3d"`
`include "MeshSurface.idp"`

```

//      build mesh of a box (311) wit 2 holes (300,310)

real hs = 0.8;
int[int] N=[4/hs,8/hs,11.5/hs];
real [int,int] B=[[-2,2],[-2,6],[-10,1.5]];
int [int,int] L=[[311,311],[311,311],[311,311]];
mesh3 ThH = SurfaceHex(N,B,L,1);
mesh3 ThSg =Sphere(1,hs,300,-1);

```

```

mesh3 ThSd =Sphere(1,hs,310,-1);    ThSd=movemesh3(ThSd,transfo=[x,4+y,z]);
mesh3 ThHS=ThH+ThSg+ThSd;           //      gluing surface meshes
medit ("ThHS", ThHS);               //      see surface mesh

real voltet=(hs^3)/6.;
real[int] domaine = [0,0,-4,1,voltet];
real [int] holes=[0,0,0,0,4,0];
mesh3 Th = tetg(ThHS,switch="pqAAYYQ",nbofregions=1,regionlist=domaine, nbofholes=2,holes=holes);
medit ("Box-With-two-Ball",Th);

//      End build mesh

int[int] opt=[9,0,64,0,0,3];         //      options of mmg3d see freeem++ doc
real[int] vit=[0,0,-0.3];
func zero = 0.;
func dep  = vit[2];

fespace Vh(Th,P1);
macro Grad(u) [dx(u),dy(u),dz(u)] //

Vh uh,vh;                          //      to compute the displacemnt field
problem Lap(uh,vh,solver=CG) = int3d(Th) (Grad(uh)'*Grad(vh)) //      ' for
emacs
+ on(310,300,uh=dep) +on(311,uh=0.);

for(int it=0; it<29; it++){
  cout<<"  ITERATION      "<<it<<endl;
  Lap;
  plot(Th,uh);
  Th=mmg3d(Th,options=opt,displacement=[zero,zero,uh],memory=1000);
}

```

5.17 A first 3d isotope mesh adaptation process

Example 5.31 (Laplace-Adapt-3d.edp)

```

load "msh3" load "tetgen" load "mshmet" load "medit"
//      build initial mesh

int nn = 6;
int[int] l1111=[1,1,1,1],l01=[0,1],l11=[1,1]; //      label numbering to have all
label to 1
mesh3 Th3=buildlayers(square(nn,nn,region=0,label=l1111),
  nn, zbound=[0,1], labelmid=l11, labelup = l01, labeldown = l01);
Th3 = trunc(Th3,(x<0.5) | (y < 0.5) | (z < 0.5),label=1); //      remove the
]0.5,1[3cube
//      end of build initial mesh

fespace Vh(Th3,P1);
Vh u,v,usol,h;

macro Grad(u) [dx(u),dy(u),dz(u)] //      EOM

problem Poisson(u,v,solver=CG) = int3d(Th3) ( Grad(u)'*Grad(v) )
-int3d(Th3) ( 1*v ) + on(1,u=0);

```



```

real errm=1e-2;                                     //    level of error
for(int ii=0; ii<5; ii++)
{
    Poisson;                                           //    solve Poisson equation.
    cout <<" u min, max = " <<  u[].min << " " << u[].max << endl;
    h=0. ;                                           //    for resizing h[] because the mesh change
    h[]=mshmet(Th3,u,normalization=1,aniso=0,nbregul=1,hmin=1e-3,hmax=0.3,err=errm);
    cout <<" h min, max = " <<  h[].min << " " << h[].max
        << " " << h[].n << " " << Th3.nv << endl;
    plot(u,wait=1);
    errm*= 0.8;                                       //    change the level of error
    cout << " Th3" << Th3.nv < " " << Th3.nt << endl;
    Th3=tetgreconstruction(Th3,switch="raAQ",sizeofvolume=h*h*h/6.); //    rebuild
    mesh
    mediat("U-adap-iso-"+ii,Th3,u,wait=1);}

```

5.18 Build a 2d mesh from a isoline

The idea is get the discretization of a isoline to fluid meshes, this tool can be useful to construct meshes from image. First, we give an example of the isovalue meshes 0.2 of analytical function $\sqrt{(x-1/2)^2 + (y-1/2)^2}$, on unit square.

Example 5.32 (isoline.edp)

```

load "isoline"                                     //    load the plugin "isoline"

real[int,int] xy(3,1);                             //    to store the isoline points
int[int] be(1);                                     //    to store the begin , end couple of lines
{                                                     //    a block for memory management
    mesh Th=square(10,10);                             //    , [x*.5,y*.5]);
    fespace Vh(Th,P1);
    Vh u= sqrt(square(x-0.5)+square(y-0.5));
    real iso= 0.2 ;
    real[int] viso=[iso];
    plot (u,viso=viso,Th);                             //    to see the iso line

    int nbc= isoline(Th,u,xy,close=1,iso=iso,beginend=be,smoothing=0.1);

```

The isoline parameters are Th the mesh, the expression u , the bidimensionnal array xy to store the list coordinate of the points. The list of named parameter are:

iso= value of the isoline to compute (0 is the default value)

close= close the iso line with the border (def. true), we add the part of the mesh border such the value is less than the iso value

smoothing= nb of smoothing process is the l's where l is the length of the current line component, r the ratio, s is smoothing value. The smoothing default value is 0.

ratio= the ratio (1 by default).

eps= relative ε (see code ??) (def 1e-10)

beginend= array to get begin, end couple of each of sub line (resize automatically)

file= to save the data curve in data file for gnu plot

In the array *xy* you get the list of vertices of the isoline, each connex line go from $i = i_0^c, \dots, i_1^c - 1$ with $i_0^c = be(2 * c)$ $i_1^c = be(2 * c + 1)$, and where $x_i = xy(0, i)$, $y_i = xy(1, i)$, $l_i = xy(2, i)$. Here l_i is the length of the line (the origin of the line is point i_0^c).

The sense of the isoline is such that the upper part is at the left size of the isoline. So here : the minimum is a point 0.5,05 so the curve 1 turn in the clockwise sense, the order of each component are sort such the the number of point by component is decreasing .

```
cout << " nb of the line component = " << nbc << endl;
cout << " n = " << xy.m << endl; // number of points
cout << "be = " << be << endl; // begin end of the each component

// show the lines component
for( int c=0; c<nbc; ++c)
{
    int i0 = be[2*c], i1 = be[2*c+1]-1; // begin,end of the line component
    cout << " Curve " << c << endl;
    for(int i=i0; i<= i1; ++i)
        cout << " x= " << xy(0,i) <<" y= " << xy(1,i) << " s= "
            << xy(2,i) << endl;
    plot([xy(0,i0:i1),xy(1,i0:i1)],wait=1,viso=viso,cmm = " curve "+c);
}
// end of block for memory management

cout << " len of last curve = " << xy(2,xy.m-1) << endl;;
```

We also have a new function to parametrize easily a discret Curve defined by couple *be,xy*.

```
border Curve0(t=0,1) // the extern boundary
{ int c = 0; // component 0
    int i0 = be[2*c], i1 = be[2*c+1]-1;
    P=Curve(xy,i0,i1,t); // Curve 0
    label=1;
}

border Curve1(t=0,1) // component 1
{ int c = 1; // component 1
    int i0 = be[2*c], i1 = be[2*c+1]-1;
    P=Curve(xy,i0,i1,t); // Curve 1
    label=1;
}

plot(Curve1(100)); // show curve.
mesh Th= buildmesh(Curve1(-100)); // because
plot(Th,wait=1); //
```

Secondly, we use this idea to build meshes from image, we use the plugins ppm2rnm to read pgm gray scale image, and we extract the gray contour at level 0.25.

Example 5.33 (Leman-mesh.edp)

```

load "ppm2rnm" load "isoline"
string leman="lg.pgm"; // see figure 5.39
real AreaLac = 580.03; // in Km^2
real hsize= 5; // mesh size in pixel ..
real[int,int] Curves(3,1);
int[int] be(1);
int nc; // nb of curve
{
  real[int,int] ffl(leman); // read image (figure 5.39)
  // and set to an rect. array
  int nx = ffl.n, ny=ffl.m; // grey value between 0 to 1 (dark)
  // build a Cartesian mesh such that the origin is at the right place.
  mesh Th=square(nx-1,ny-1,[(nx-1)*(x),(ny-1)*(1-y)]);
  // warning the numbering is of the vertices (x,y) is
  // given by i=x/nx+nx*y/ny

  fespace Vh(Th,P1);
  Vh f1; f1[]=ffl; // transforme array in finite element function.
  nc=isoline(Th,f1,iso=0.25,close=1,Curves,beginend=be,smoothing=.1, ratio=0.5);
}

// the longest isoline : the lac ..
int ic0=be(0), ic1=be(1)-1;
plot([Curves(0,ic0:ic1),Curves(1,ic0:ic1)], wait=1);
int NC= Curves(2,ic1)/hsize;
border G(t=0,1) { P=Curve(Curves,ic0,ic1,t); label= 1 + (x>xl)*2 + (y<yl);}

plot(G(-NC),wait=1);
mesh Th=buildmesh(G(-NC));
plot(Th,wait=1);
real scale = sqrt(AreaLac/Th.area);
Th=movemesh(Th,[x*scale,y*scale]); // resize the mesh
cout << " Th.area = " << Th.area << " Km^2 " << " == " << AreaLac << " Km^2 "
<< endl ;
plot(Th,wait=1,ps="leman.eps"); // see figure 5.40

```

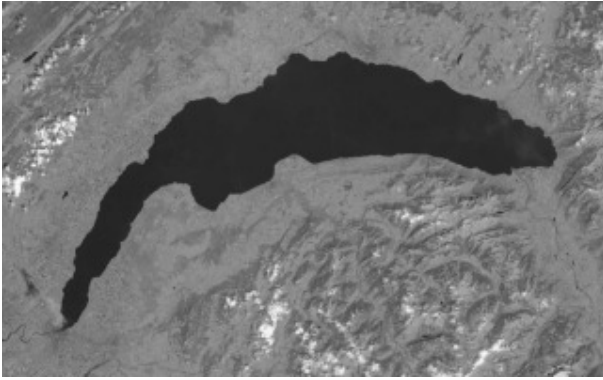


Figure 5.39: The image of the leman lac meshes

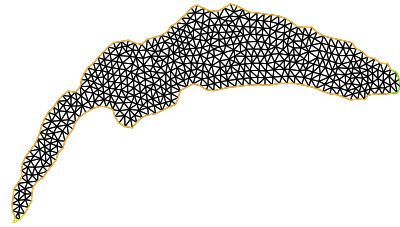


Figure 5.40: the mesh of lac

Chapter 6

Finite Elements

As stated in Section 2. FEM approximates all functions w as

$$w(x, y) \simeq w_0\phi_0(x, y) + w_1\phi_1(x, y) + \cdots + w_{M-1}\phi_{M-1}(x, y)$$

with finite element basis functions $\phi_k(x, y)$ and numbers w_k ($k = 0, \dots, M - 1$). The functions $\phi_k(x, y)$ are constructed from the triangle T_{i_k} , and called *shape functions*. In FreeFem++ the finite element space

$$V_h = \{w \mid w_0\phi_0 + w_1\phi_1 + \cdots + w_{M-1}\phi_{M-1}, w_i \in \mathbb{R}\}$$

is easily created by

```
fespace IDspace (IDmesh, <IDFE>) ;
```

or with ℓ pairs of periodic boundary condition in 2d

```
fespace IDspace (IDmesh, <IDFE>,
                  periodic=[ [la_1, sa_1], [lb_1, sb_1],
                              ...
                              [la_k, sa_k], [lb_k, sb_k] ] );
```

and in 3d

```
fespace IDspace (IDmesh, <IDFE>,
                  periodic=[ [la_1, sa_1, ta_1], [lb_1, sb_1, tb_1],
                              ...
                              [la_k, sa_k, ta_k], [lb_k, sb_k, tb_k] ] );
```

where

IDspace is the name of the space (e.g. V_h),

IDmesh is the name of the associated mesh and <IDFE> is a identifier of finite element type.

In 2D we have a pair of periodic boundary condition, if $[la_i, sa_i], [lb_i, sb_i]$ is a pair of int, and the 2 labels la_i and lb_i refer to 2 pieces of boundary to be in equivalence.

If $[la_i, sa_i], [lb_i, sb_i]$ is a pair of real, then sa_i and sb_i give two common abscissa on the two boundary curve, and two points are identified as one if the two abscissa are equal.

In 2D, we have a pair of periodic boundary condition, if $[la_i, sa_i, ta_i], [lb_i, sb_i, tb_i]$ is a pair of int, the 2 labels la_i and lb_i define the 2 piece of boundary to be in equivalence.

If $[la_i, sa_i, ta_i], [lb_i, sb_i, tb_i]$ is a pair of real, then sa_i, ta_i and sb_i, tb_i give two common parameters on the two boundary surface, and two points are identified as one if the two parameters are equal.

Remark 6 *The 2D mesh of the two identified borders must be the same, so to be sure, use the parameter `fixeborder=true` in `buildmesh` command (see 5.1.2) like in example `periodic2bis.edp` (see 9.7).*

As of today, the known types of finite element are:

P0,P03d piecewise constant discontinuous finite element (2d, 3d), the degrees of freedom are the barycenter element value.

$$P0_h = \{v \in L^2(\Omega) \mid \text{for all } K \in \mathcal{T}_h \text{ there is } \alpha_K \in \mathbb{R} : v|_K = \alpha_K\} \quad (6.1)$$

P1,P13d piecewise linear continuous finite element (2d, 3d), the degrees of freedom are the vertices values.

$$P1_h = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1\} \quad (6.2)$$

P1dc piecewise linear discontinuous finite element

$$P1dc_h = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1\} \quad (6.3)$$

Warning, due to interpolation problem, the degree of freedom is not the vertices but three vertices move inside with $T(X) = G + .99(X - G)$ where G is the barycenter, (version 2.24-4).

P1b,P1b3d piecewise linear continuous finite element plus bubble (2d, 3d)

The 2d case:

$$P1b_h = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1 \oplus \text{Span}\{\lambda_0^K \lambda_1^K \lambda_2^K\}\} \quad (6.4)$$

The 3d case:

$$P1b_h = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1 \oplus \text{Span}\{\lambda_0^K \lambda_1^K \lambda_2^K \lambda_3^K\}\} \quad (6.5)$$

where $\lambda_i^K, i = 0, \dots, d$ are the $d+1$ barycentric coordinate functions of the element K (triangle or tetrahedron).

P2,P23d piecewise P_2 continuous finite element (2d, 3d),

$$P2_h = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2\} \quad (6.6)$$

where P_2 is the set of polynomials of \mathbb{R}^2 of degrees ≤ 2 .

P2b piecewise P_2 continuous finite element plus bubble,

$$P2_h = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2 \oplus \text{Span}\{\lambda_0^K \lambda_1^K \lambda_2^K\}\} \quad (6.7)$$

P2dc piecewise P_2 discontinuous finite element,

$$P2dc_h = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2\} \quad (6.8)$$

Warning, due to interpolation problem, the degree of freedom is not the six P2 nodes but six nodes move inside with $T(X) = G + .99(X - G)$ where G is the barycenter, (version 2.24-4).

P3 piecewise P_3 continuous finite element (2d) (need `load "Element_P3"`)

$$P2_h = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_3\} \quad (6.9)$$

where P_3 is the set of polynomials of \mathbb{R}^2 of degrees ≤ 3 .

P3dc piecewise P_3 discontinuous finite element (2d) (need `load "Element_P3dc"`)

$$P2_h = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_3\} \quad (6.10)$$

where P_3 is the set of polynomials of \mathbb{R}^2 of degrees ≤ 3 .

P4 piecewise P_4 continuous finite element (2d) (need `load "Element_P4"`)

$$P2_h = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_4\} \quad (6.11)$$

where P_4 is the set of polynomials of \mathbb{R}^2 of degrees ≤ 4 .

P4dc piecewise P_4 discontinuous finite element (2d) (need `load "Element_P4dc"`)

$$P2_h = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_3\} \quad (6.12)$$

where P_4 is the set of polynomials of \mathbb{R}^2 of degrees ≤ 3 .

P0Edge piecewise P_0 discontinuous finite element (2d) constant on each edge of the mesh.

P1Edge piecewise P_1 discontinuous finite element (2d) (need `load "Element_PkEdge"`) P_1 on each edge of the mesh.

P2Edge piecewise P_2 discontinuous finite element (2d) (need `load "Element_PkEdge"`) P_2 on each edge of the mesh.

P3Edge piecewise P_3 discontinuous finite element (2d) (need `load "Element_PkEdge"`) P_3 on each edge of the mesh.

P4Edge piecewise P_4 discontinuous finite element (2d) (need `load "Element_PkEdge"`) P_4 on each edge of the mesh.

P5Edge piecewise P_5 discontinuous finite element (2d) (need `load "Element_PkEdge"`) P_5 on each edge of the mesh.

Morley piecewise P_2 non conform finite element (2d) (need `load "Morley"`)

$$P2_h = \left\{ v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_3, \left\{ \begin{array}{l} v \text{ continuous at vertices,} \\ \partial_n v \text{ continuous at middle of edge,} \end{array} \right. \right\} \quad (6.13)$$

where P_2 is the set of polynomials of \mathbb{R}^2 of degrees ≤ 3 .

Warning to build the interpolant of a function u (scalar) for this finite element, we need the function and 2 partial derivatives (u, u_x, u_y) , so this vectorial finite element with 3 components (u, u_x, u_y) .

See example `bilapMorley.edp` of `examples++-load` for solving BiLaplacien problem :

```

load "Morley"
fespace Vh(Th,P2Morley);           // the Morley finite element space
macro bilaplacien(u,v) ( dxx(u)*dxx(v)+dyy(u)*dyy(v)+2.*dxy(u)*dxy(v) )
// fin macro

real f=1;
Vh [u,ux,uy],[v,vx,vy];

solve bilap([u,ux,uy],[v,vx,vy]) =
  int2d(Th) ( bilaplacien(u,v) )
- int2d(Th) ( f*v )
+ on(1,2,3,4,u=0,ux=0,uy=0)

```

P2BR (need `load "BernadiRaugel"`) the Bernadi Raugel Finite Element is a Vectorial element (2d) with 2 components, See Bernardi, C., Raugel, G.: Analysis of some finite elements for the Stokes problem. Math. Comp. 44, 71-79 (1985). It is a 2d coupled FE, with the Polynomial space is $P_1^2 + 3$ normals bubbles edges function (P_2) and the degree of freedom is 6 values at of the 2 components at the 3 vertices and the 3 flux on the 3 edges So the number degrees of freedom is 9.

RT0,RT03d Raviart-Thomas finite element of degree 0.

The 2d case:

$$RT0_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 \\ \alpha_K^2 \end{vmatrix} + \beta_K \begin{vmatrix} x \\ y \end{vmatrix} \right\} \quad (6.14)$$

The 3d case:

$$RT0_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y, z) = \begin{vmatrix} \alpha_K^1 \\ \alpha_K^2 \\ \alpha_K^3 \end{vmatrix} + \beta_K \begin{vmatrix} x \\ y \\ z \end{vmatrix} \right\} \quad (6.15)$$

where by writing $\text{div } \mathbf{w} = \sum_{i=1}^d \partial w_i / \partial x_i$ with $\mathbf{w} = (w_i)_{i=1}^d$,

$$H(\text{div}) = \left\{ \mathbf{w} \in L^2(\Omega)^d \mid \text{div } \mathbf{w} \in L^2(\Omega) \right\}$$

and where $\alpha_K^1, \alpha_K^2, \alpha_K^3, \beta_K$ are real numbers.

RT0Ortho Raviart-Thomas Orthogonal, or Nedelec finite element type I of degree 0 in dimension 2

$$RT0Ortho_h = \left\{ \mathbf{v} \in H(\text{curl}) \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 \\ \alpha_K^2 \end{vmatrix} + \beta_K \begin{vmatrix} -y \\ x \end{vmatrix} \right\} \quad (6.16)$$

Edge03d Nedelec finite element or Edge Element of degree 0.

The 3d case:

$$Edge0_h = \left\{ \mathbf{v} \in H(\text{Curl}) \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y, z) = \begin{vmatrix} \alpha_K^1 \\ \alpha_K^2 \\ \alpha_K^3 \end{vmatrix} + \begin{vmatrix} \beta_K^1 \\ \beta_K^2 \\ \beta_K^3 \end{vmatrix} \times \begin{vmatrix} x \\ y \\ z \end{vmatrix} \right\} \quad (6.17)$$

where by writing $\text{curl } \mathbf{w} = \begin{vmatrix} \partial w_2 / \partial x_3 - \partial w_3 / \partial x_2 \\ \partial w_3 / \partial x_1 - \partial w_1 / \partial x_3 \\ \partial w_1 / \partial x_2 - \partial w_2 / \partial x_1 \end{vmatrix}$ with $\mathbf{w} = (w_i)_{i=1}^d$,

$$H(\text{curl}) = \left\{ \mathbf{w} \in L^2(\Omega)^d \mid \text{curl } \mathbf{w} \in L^2(\Omega)^d \right\}$$

and $\alpha_K^1, \alpha_K^2, \alpha_K^3, \beta_K^1, \beta_K^2, \beta_K^3$ are real numbers.

P1nc piecewise linear element continuous at the middle of edge only in 2D ?????.

RT1 (need `load "Element_Mixte"`, version 3.13)

$$RT1_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h \quad (\alpha_K^2, \alpha_K^2, \beta_K) \in P_1^3, \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 \\ \alpha_K^2 \end{vmatrix} + \beta_K \begin{vmatrix} x \\ y \end{vmatrix} \right\} \quad (6.18)$$

RT1Ortho (need `load "Element_Mixte"`, version 3.13, dimension 2)

$$RT1_h = \left\{ \mathbf{v} \in H(\text{curl}) \mid \forall K \in \mathcal{T}_h \quad (\alpha_K^2, \alpha_K^2, \beta_K) \in P_1^3, \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 \\ \alpha_K^2 \end{vmatrix} + \beta_K \begin{vmatrix} -y \\ x \end{vmatrix} \right\} \quad (6.19)$$

BDM1 (need `load "Element_Mixte"`, version 3.13, dimension 2) the Brezzi-Douglas-Marini finite element

$$BDM1_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K \in P_1^2 \right\} \quad (6.20)$$

BDM1Ortho (need `load "Element_Mixte"`, version 3.13, dimension 2) the Brezzi-Douglas-Marini Orthogonal also call Nedelec of type II , finite element

$$BDM1Ortho_h = \left\{ \mathbf{v} \in H(\text{curl}) \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K \in P_1^2 \right\} \quad (6.21)$$

TDNNS1 (need `load "Element_Mixte"`, version 3.13, dimension 2) A new element finite element to approximation symetrique 2x2 matrix in $H(\text{divdiv})$ (i.e σ_{nn} is continuous across edge).

$$TDNNS1_h = \left\{ \boldsymbol{\sigma} \in (L^2)^{2,2} \mid \forall K \in \mathcal{T}_h \quad \boldsymbol{\sigma}|_K \in P_1^2, \sigma_{12} = \sigma_{21}, \sigma_{nn} \text{ is continuous} \right\} \quad (6.22)$$

where $\sigma_{nn} = n^t \sigma n$, and n is a normal to the edge (see [41, section 4.2.2.3] for full detail)

6.1 Use of “fespace” in 2d

With the 2d finite element spaces

$$X_h = \{v \in H^1([0, 1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1\}$$

$$X_{ph} = \{v \in X_h \mid v(\begin{vmatrix} 0 \\ 1 \end{vmatrix}) = v(\begin{vmatrix} 1 \\ 1 \end{vmatrix}), v(\begin{vmatrix} 0 \\ 0 \end{vmatrix}) = v(\begin{vmatrix} 1 \\ 0 \end{vmatrix})\}$$

$$M_h = \{v \in H^1([0, 1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2\}$$

$$R_h = \{\mathbf{v} \in H^1([0, 1]^2)^2 \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K \\ \beta_K \end{vmatrix} + \gamma_K \begin{vmatrix} x \\ y \end{vmatrix}\}$$

when \mathcal{T}_h is a mesh 10×10 of the unit square $]0, 1[^2$, we only write in FreeFem++ :

```
mesh Th=square(10,10);
fespace Xh(Th,P1); // scalar FE
fespace Xph(Th,P1,
    periodic=[[2,y],[4,y],[1,x],[3,x]]); // bi-periodic FE
fespace Mh(Th,P2); // scalar FE
fespace Rh(Th,RT0); // vectorial FE
```

where X_h, M_h, R_h expresses finite element spaces (called FE spaces) X_h, M_h, R_h , respectively.

To use FE-functions $u_h, v_h \in X_h, p_h, q_h \in M_h$ and $U_h, V_h \in R_h$, we write :

```

Xh uh,vh;
Xph uph,vph;
Mh ph,qh;
Rh [Uxh,Uyh],[Vxh,Vyh];
Xh[int] Uh(10); // array of 10 function in Xh
Rh[int] [Wxh,Wyh](10); // array of 10 functions in Rh.
Wxh[5](0.5,0.5) // the 6th function at point (0.5,0.5)
Wxh[5][] // the array of the degree of freedom of the 6 function.

```

The functions U_h, V_h have two components so we have

$$U_h = \begin{pmatrix} U_{xh} \\ U_{yh} \end{pmatrix} \quad \text{and} \quad V_h = \begin{pmatrix} V_{xh} \\ V_{yh} \end{pmatrix}$$

6.2 Use of fespace in 3d

With the 3d finite element spaces

$$X_h = \{v \in H^1([0,1]^3) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1\}$$

$$X_{ph} = \{v \in X_h \mid v(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}) = v(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}), v(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}) = v(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})\}$$

$$M_h = \{v \in H^1([0,1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2\}$$

$$R_h = \{v \in H^1([0,1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K(x,y) = \begin{pmatrix} \alpha_K \\ \beta_K \end{pmatrix} + \gamma_K \begin{pmatrix} x \\ y \end{pmatrix}\}$$

when \mathcal{T}_h is a mesh $10 \times 10 \times 10$ of the unit cubic $]0,1[^2$, we write in FreeFem++ :

```

mesh3 Th=buildlayers(square(10,10),10, zbound=[0,1]);
// label: 0 up, 1 down; 2 front, 3 left, 4 back, 5: right
fespace Xh(Th,P1); // scalar FE
fespace Xph(Th,P1,
    periodic=[ [0,x,y],[1,x,y],
               [2,x,z],[4,x,z],
               [3,y,z],[5,y,z] ]); // three-periodic FE (see Note 6.1)
fespace Mh(Th,P2); // scalar FE
fespace Rh(Th,RT03d); // vectorial FE

```

where X_h, M_h, R_h expresses finite element spaces (called FE spaces) X_h, M_h, R_h , respectively.

To define and use FE-functions $u_h, v_h \in X_h$ and $p_h, q_h \in M_h$ and $U_h, V_h \in R_h$, we write:

```

Xh uh,vh;
Xph uph,vph;
Mh ph,qh;
Rh [Uxh,Uyh,Uyzh],[Vxh,Vyh,Vyzh];
Xh[int] Uh(10); // array of 10 function in Xh
Rh[int] [Wxh,Wyh,Wzh](10); // array of 10 functions in Rh.
Wxh[5](0.5,0.5,0.5) // the 6th function at point (0.5,0.5,0.5)
Wxh[5][] // the array of the degree of
freedom of the 6 function.

```

The functions U_h, V_h have three components so we have

$$U_h = \begin{pmatrix} U_{xh} \\ U_{yh} \\ U_{zh} \end{pmatrix} \quad \text{and} \quad V_h = \begin{pmatrix} V_{xh} \\ V_{yh} \\ V_{zh} \end{pmatrix}$$

Note 6.1 *One hard problem of the periodic boundary condition is the mesh must be the same on equivalence face, the BuildLayer mesh generator split each quadrilateral faces with the diagonal passing through vertex with maximal number, so to be sure to have the same mesh on both face periodic the 2d numbering in corresponding edges must be compatible (for example the same variation). By Default, the numbering of square vertex are correct.*

To change the mesh numbering you can use the change function like:

```
{
//      for cleaning memory..
int[int] old2new(0:Th.nv-1);      //      array set on 0, 1, ..., nv-1
fespace Vh2(Th,P1);
Vh2 sorder=x+y; //      choose an ordering increasing on 4 square borders with x
or y
sort(sorder[],old2new);           //      build the inverse permutation
int[int] new2old=old2new^-1;      //      inverse the permutation
Th= change(Th,renumv=new2old);change
}
```

the full example is in examples++-3d/periodic-3d.edp

6.3 Lagrangian Finite Elements

6.3.1 P0-element

For each triangle ($d=2$) or tetrahedron ($d=3$) T_k , the basis function ϕ_k in $Vh(Th, P0)$ is given by

$$\phi_k(\mathbf{x}) = 1 \text{ if } (\mathbf{x}) \in T_k, \quad \phi_k(\mathbf{x}) = 0 \text{ if } (\mathbf{x}) \notin T_k$$

If we write

$Vh(Th, P0); \quad Vh \text{ fh} = f(x, y);$

then for vertices q^{k_i} , $i = 1, 2, \dots, d+1$ in Fig. 6.1(a), f_h is built as

$$fh = f_h(x, y) = \sum_k f\left(\frac{\sum_i q^{k_i}}{d+1}\right) \phi_k$$

See Fig. 6.3 for the projection of $f(x, y) = \sin(\pi x) \cos(\pi y)$ on $Vh(Th, P0)$ when the mesh Th is a 4×4 -grid of $[-1, 1]^2$ as in Fig. 6.2.

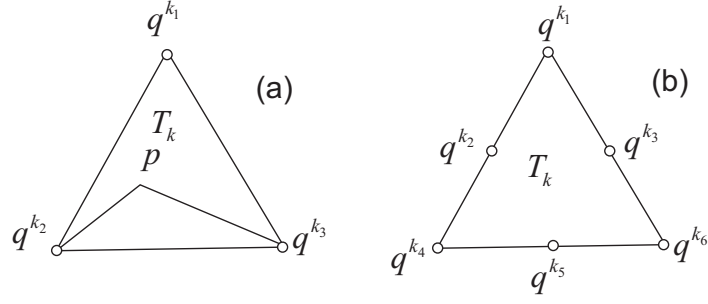
6.3.2 P1-element

For each vertex q^i , the basis function ϕ_i in $Vh(Th, P1)$ is given by

$$\begin{aligned} \phi_i(x, y) &= a_i^k + b_i^k x + c_i^k y \text{ for } (x, y) \in T_k, \\ \phi_i(q^i) &= 1, \quad \phi_i(q^j) = 0 \text{ if } i \neq j \end{aligned}$$

The basis function $\phi_{k_1}(x, y)$ with the vertex q^{k_1} in Fig. 6.1(a) at point $p = (x, y)$ in triangle T_k simply coincide with the *barycentric coordinates* λ_1^k (*area coordinates*) :

$$\phi_{k_1}(x, y) = \lambda_1^k(x, y) = \frac{\text{area of triangle}(p, q^{k_2}, q^{k_3})}{\text{area of triangle}(q^{k_1}, q^{k_2}, q^{k_3})}$$

Figure 6.1: P_1 and P_2 degrees of freedom on triangle T_k

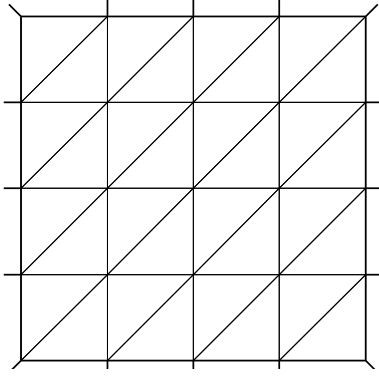
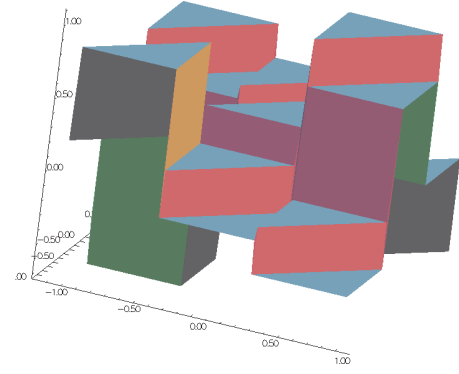
If we write

$\text{Vh}(\text{Th}, \mathbf{P1})$; $\text{Vh fh} = g(x, y)$;

then

$$\text{fh} = f_h(x, y) = \sum_{i=1}^{n_v} f(q^i) \phi_i(x, y)$$

See Fig. 6.4 for the projection of $f(x, y) = \sin(\pi x) \cos(\pi y)$ into $\text{Vh}(\text{Th}, \mathbf{P1})$.

Figure 6.2: Test mesh Th for projectionFigure 6.3: projection to $\text{Vh}(\text{Th}, \mathbf{P0})$

6.3.3 P2-element

For each vertex or midpoint q^i , the basis function ϕ_i in $\text{Vh}(\text{Th}, \mathbf{P2})$ is given by

$$\begin{aligned} \phi_i(x, y) &= a_i^k + b_i^k x + c_i^k y + d_i^k x^2 + e_i^k xy + f_j^k y^2 \text{ for } (x, y) \in T_k, \\ \phi_i(q^i) &= 1, \quad \phi_i(q^j) = 0 \text{ if } i \neq j \end{aligned}$$

The basis function $\phi_{k_1}(x, y)$ with the vertex q^{k_1} in Fig. 6.1(b) is defined by the *barycentric coordinates*:

$$\phi_{k_1}(x, y) = \lambda_1^k(x, y)(2\lambda_1^k(x, y) - 1)$$

and for the midpoint q^{k_2}

$$\phi_{k_2}(x, y) = 4\lambda_1^k(x, y)\lambda_4^k(x, y)$$

If we write

$$\text{Vh}(\text{Th}, \mathbf{P2}); \quad \text{Vh} \quad \text{fh} = f(x, y);$$

then

$$\text{fh} = f_h(x, y) = \sum_{i=1}^M f(q^i) \phi_i(x, y) \quad (\text{summation over all vetex or midpoint})$$

See Fig. 6.5 for the projection of $f(x, y) = \sin(\pi x) \cos(\pi y)$ into $\text{Vh}(\text{Th}, \mathbf{P2})$.

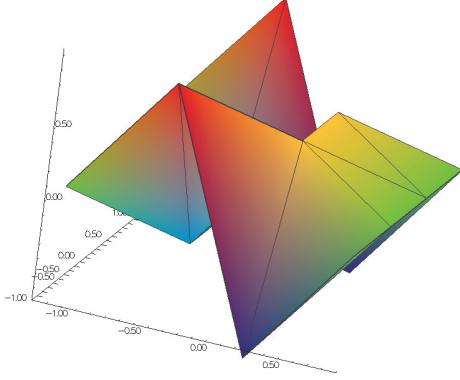


Figure 6.4: projection to $\text{Vh}(\text{Th}, \mathbf{P1})$

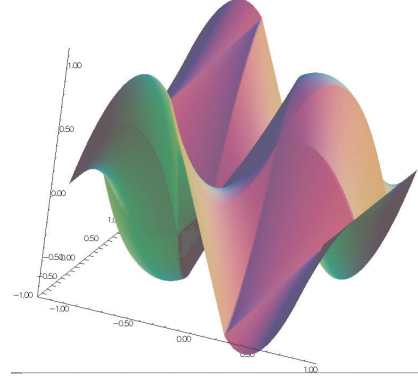


Figure 6.5: projection to $\text{Vh}(\text{Th}, \mathbf{P2})$

6.4 P1 Nonconforming Element

Refer to [23] for details; briefly, we now consider non-continuous approximations so we shall lose the property

$$w_h \in V_h \subset H^1(\Omega)$$

If we write

$$\text{Vh}(\text{Th}, \mathbf{P1nc}); \quad \text{Vh} \quad \text{fh} = f(x, y);$$

then

$$\text{fh} = f_h(x, y) = \sum_{i=1}^{n_v} f(m^i) \phi_i(x, y) \quad (\text{summation over all midpoint})$$

Here the basis function ϕ_i associated with the midpoint $m^i = (q^{k_i} + q^{k_{i+1}})/2$ where q^{k_i} is the i -th point in T_k , and we assume that $j + 1 = 0$ if $j = 3$:

$$\begin{aligned} \phi_i(x, y) &= a_i^k + b_i^k x + c_i^k y \quad \text{for } (x, y) \in T_k, \\ \phi_i(m^i) &= 1, \quad \phi_i(m^j) = 0 \quad \text{if } i \neq j \end{aligned}$$

Strictly speaking $\partial \phi_i / \partial x$, $\partial \phi_i / \partial y$ contain Dirac distribution $\rho \delta_{\partial T_k}$. The numerical calculations will automatically *ignore* them. In [23], there is a proof of the estimation

$$\left(\sum_{k=1}^{n_v} \int_{T_k} |\nabla w - \nabla w_h|^2 dx dy \right)^{1/2} = O(h)$$

The basis functions ϕ_k have the following properties.

1. For the bilinear form a defined in (2.6) satisfy

$$a(\phi_i, \phi_i) > 0, \quad a(\phi_i, \phi_j) \leq 0 \quad \text{if } i \neq j$$

$$\sum_{k=1}^{n_v} a(\phi_i, \phi_k) \geq 0$$

2. $f \geq 0 \Rightarrow u_h \geq 0$

3. If $i \neq j$, the basis function ϕ_i and ϕ_j are L^2 -orthogonal:

$$\int_{\Omega} \phi_i \phi_j \, dx dy = 0 \quad \text{if } i \neq j$$

which is false for P_1 -element.

See Fig. 6.6 for the projection of $f(x, y) = \sin(\pi x) \cos(\pi y)$ into $V_h(\mathcal{T}_h, \mathbf{P1nc})$. See Fig. 6.6 for the projection of $f(x, y) = \sin(\pi x) \cos(\pi y)$ into $V_h(\mathcal{T}_h, \mathbf{P1nc})$.

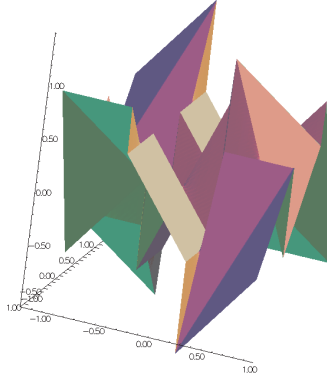


Figure 6.6: projection to $V_h(\mathcal{T}_h, \mathbf{P1nc})$

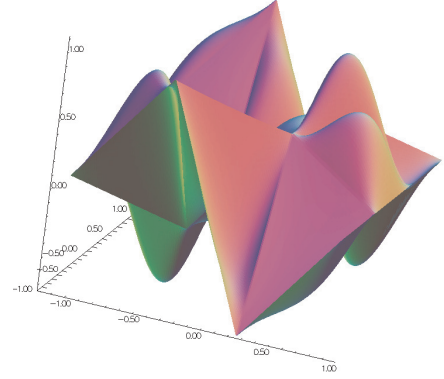


Figure 6.7: projection to $V_h(\mathcal{T}_h, \mathbf{P1b})$

6.5 Other FE-space

For each triangle $T_k \in \mathcal{T}_h$, let $\lambda_{k_1}(x, y)$, $\lambda_{k_2}(x, y)$, $\lambda_{k_3}(x, y)$ be the area coordinate of the triangle (see Fig. 6.1), and put

$$\beta_k(x, y) = 27\lambda_{k_1}(x, y)\lambda_{k_2}(x, y)\lambda_{k_3}(x, y) \quad (6.23)$$

called *bubble* function on T_k . The bubble function has the feature:

1. $\beta_k(x, y) = 0$ if $(x, y) \in \partial T_k$.
2. $\beta_k(q^{k_b}) = 1$ where q^{k_b} is the barycenter $\frac{q^{k_1} + q^{k_2} + q^{k_3}}{3}$.

If we write

$$V_h(\mathcal{T}_h, \mathbf{P1b}); \quad V_h \text{ fh} = f(x, y);$$

then

$$\text{fh} = f_h(x, y) = \sum_{i=1}^{n_v} f(q^i) \phi_i(x, y) + \sum_{k=1}^{n_t} f(q^{k_b}) \beta_k(x, y)$$

See Fig. 6.7 for the projection of $f(x, y) = \sin(\pi x) \cos(\pi y)$ into $V_h(\mathcal{T}_h, \mathbf{P1b})$.

6.6 Vector valued FE-function

Functions from \mathbb{R}^2 to \mathbb{R}^N with $N = 1$ is called scalar function and called *vector valued* when $N > 1$. When $N = 2$

fespace Vh (Th, [P0, P1]) ;

make the space

$$V_h = \{\mathbf{w} = (w_1, w_2) \mid w_1 \in V_h(\mathcal{T}_h, P_0), w_2 \in V_h(\mathcal{T}_h, P_1)\}$$

6.6.1 Raviart-Thomas element

In the Raviart-Thomas finite element $RT0_h$, the degree of freedom are the fluxes across edges e of the mesh, where the flux of the function $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is $\int_e \mathbf{f} \cdot \mathbf{n}_e$, \mathbf{n}_e is the unit normal of edge e . This implies a orientation of all the edges of the mesh, for example we can use the global numbering of the edge vertices and we just go from small to large numbers.

To compute the flux, we use a quadrature with one Gauss point, the middle point of the edge. Consider a triangle T_k with three vertices $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Let denote the vertices numbers by i_a, i_b, i_c , and define the three edge vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ by $\text{sgn}(i_b - i_c)(\mathbf{b} - \mathbf{c})$, $\text{sgn}(i_c - i_a)(\mathbf{c} - \mathbf{a})$, $\text{sgn}(i_a - i_b)(\mathbf{a} - \mathbf{b})$, We get three basis functions,

$$\phi_1^k = \frac{\text{sgn}(i_b - i_c)}{2|T_k|}(\mathbf{x} - \mathbf{a}), \quad \phi_2^k = \frac{\text{sgn}(i_c - i_a)}{2|T_k|}(\mathbf{x} - \mathbf{b}), \quad \phi_3^k = \frac{\text{sgn}(i_a - i_b)}{2|T_k|}(\mathbf{x} - \mathbf{c}), \quad (6.24)$$

where $|T_k|$ is the area of the triangle T_k . If we write

Vh (Th, RT0) ; Vh [f1h, f2h] = [f1(x,y), f2(x,y)] ;

then

$$\mathbf{f}_h = \mathbf{f}_h(x, y) = \sum_{k=1}^{n_t} \sum_{l=1}^6 n_{i_l j_l} |\mathbf{e}^{i_l}| f_{j_l}(m^{i_l}) \phi_{i_l j_l}$$

where $n_{i_l j_l}$ is the j_l -th component of the normal vector \mathbf{n}_{i_l} ,

$$\{m_1, m_2, m_3\} = \left\{ \frac{\mathbf{b} + \mathbf{c}}{2}, \frac{\mathbf{a} + \mathbf{c}}{2}, \frac{\mathbf{b} + \mathbf{a}}{2} \right\}$$

and $i_l = \{1, 1, 2, 2, 3, 3\}$, $j_l = \{1, 2, 1, 2, 1, 2\}$ with the order of l .

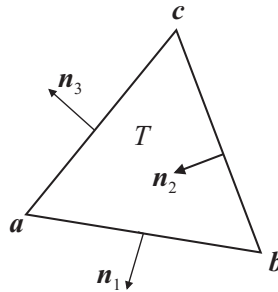
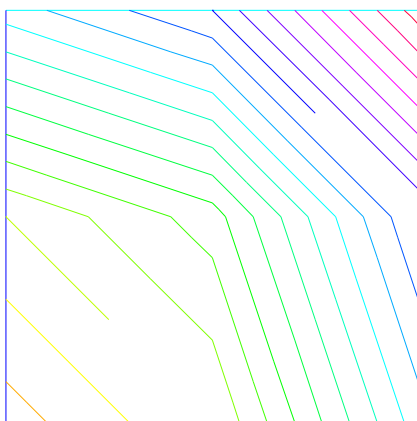
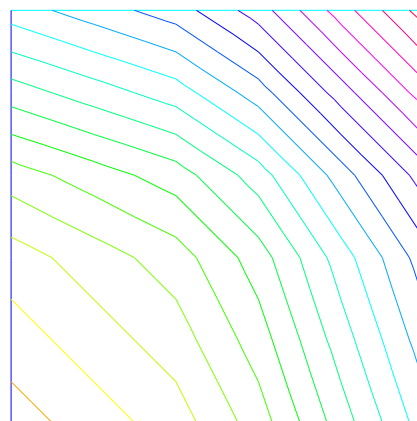


Figure 6.8: normal vectors of each edge

Example 6.1 `mesh` `Th=square(2,2);`
`fespace` `Xh(Th,P1);`
`fespace` `Vh(Th,RT0);`
`Xh` `uh,vh;`
`Vh` `[Uxh,Uyh];`
`[Uxh,Uyh] = [sin(x),cos(y)];` *// ok vectorial FE function*
`vh= x^2+y^2;` *// vh*
`Th = square(5,5);` *// change the mesh*
// Xh is unchange
// compute on the new Xh
`uh = x^2+y^2;` *// error: impossible to set only 1 component*
`Uxh = x;` *// of a vector FE function.*
`vh = Uxh;` *// ok*
// and now uh use the 5x5 mesh
// but the fespace of vh is always the 2x2 mesh
`plot(uh,ps="onoldmesh.eps");` *// figure 6.9*
`uh = uh;` *// do a interpolation of vh (old) of 5x5 mesh*
// to get the new vh on 10x10 mesh.
`plot(uh,ps="onnewmesh.eps");` *// figure 6.10*
`vh([x-1/2,y])= x^2 + y^2;` *// interpolate vh = ((x-1/2)^2 + y^2)*

Figure 6.9: `vh` Iso on mesh 2×2 Figure 6.10: `vh` Iso on mesh 5×5

To get the value at a point $x = 1, y = 2$ of the FE function `uh`, or `[Uxh,Uyh]`, one writes

```
real value;
value = uh(2,4);           // get value= uh(2,4)
value = Uxh(2,4);         // get value= Uxh(2,4)
                           // ----- or -----
x=1;y=2;
value = uh;               // get value= uh(1,2)
value = Uxh;              // get value= Uxh(1,2)
value = Uyh;              // get value= Uyh(1,2).
```

To get the value of the array associated to the FE function `uh`, one writes


```

real value = uh[][0] ;           //    get the value of degree of freedom 0
real maxdf = uh[].max;           //    maximum value of degree of freedom
int size = uh.n;                 //    the number of degree of freedom
real[int] array(uh.n)= uh[];    //    copy the array of the function uh

```

Note 6.2 For a none scalar finite element function $[Uxh, Uy_h]$ the two array $Uxh[]$ and $Uy_h[]$ are the same array, because the degree of freedom can touch more than one component.

6.7 A Fast Finite Element Interpolator

In practice one may discretize the variational equations by the Finite Element method. Then there will be one mesh for Ω_1 and another one for Ω_2 . The computation of integrals of products of functions defined on different meshes is difficult. Quadrature formulae and interpolations from one mesh to another at quadrature points are needed. We present below the interpolation operator which we have used and which is new, to the best of our knowledge. Let $\mathcal{T}_h^0 = \cup_k T_k^0$, $\mathcal{T}_h^1 = \cup_k T_k^1$

be two triangulations of a domain Ω . Let

$$V(\mathcal{T}_h^i) = \{C^0(\Omega_h^i) : f|_{T_k^i} \in P_0\}, \quad i = 0, 1$$

be the spaces of continuous piecewise affine functions on each triangulation.

Let $f \in V(\mathcal{T}_h^0)$. The problem is to find $g \in V(\mathcal{T}_h^1)$ such that

$$g(q) = f(q) \quad \forall q \text{ vertex of } \mathcal{T}_h^1$$

Although this is a seemingly simple problem, it is difficult to find an efficient algorithm in practice. We propose an algorithm which is of complexity $N^1 \log N^0$, where N^i is the number of vertices of \mathcal{T}_h^i , and which is very fast for most practical 2D applications.

Algorithm

The method has 5 steps. First a quadtree is built containing all the vertices of mesh \mathcal{T}_h^0 such that in each terminal cell there are at least one, and at most 4, vertices of \mathcal{T}_h^0 .

For each q^1 , vertex of \mathcal{T}_h^1 do:

Step 1 Find the terminal cell of the quadtree containing q^1 .

Step 2 Find the the nearest vertex q_j^0 to q^1 in that cell.

Step 3 Choose one triangle $T_k^0 \in \mathcal{T}_h^0$ which has q_j^0 for vertex.

Step 4 Compute the barycentric coordinates $\{\lambda_j\}_{j=1,2,3}$ of q^1 in T_k^0 .

- – if all barycentric coordinates are positive, go to Step 5
- – else if one barycentric coordinate λ_i is negative replace T_k^0 by the adjacent triangle opposite q_i^0 and go to Step 4.
- – else two barycentric coordinates are negative so take one of the two randomly and replace T_k^0 by the adjacent triangle as above.

Step 5 Calculate $g(q^1)$ on T_k^0 by linear interpolation of f :

$$g(q^1) = \sum_{j=1,2,3} \lambda_j f(q_j^0)$$

End

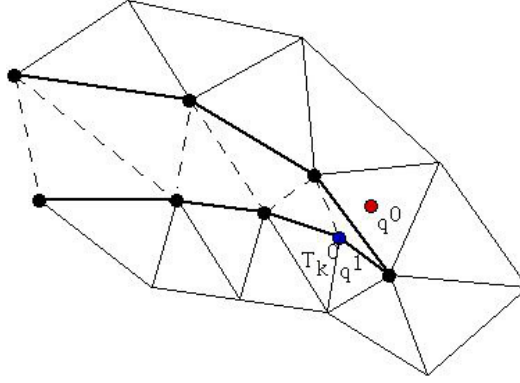


Figure 6.11: To interpolate a function at q^0 the knowledge of the triangle which contains q^0 is needed. The algorithm may start at $q^1 \in T_k^0$ and stall on the boundary (thick line) because the line $q^0 q^1$ is not inside Ω . But if the holes are triangulated too (doted line) then the problem does not arise.

Two problems need to be solved:

- What if q^1 is not in Ω_h^0 ? Then Step 5 will stop with a boundary triangle. So we add a step which test the distance of q^1 with the two adjacent boundary edges and select the nearest, and so on till the distance grows.
- What if Ω_h^0 is not convex and the marching process of Step 4 locks on a boundary? By construction Delaunay-Voronoi mesh generators always triangulate the convex hull of the vertices of the domain. So we make sure that this information is not lost when $\mathcal{T}_h^0, \mathcal{T}_h^1$ are constructed and we keep the triangles which are outside the domain in a special list. Hence in step 5 we can use that list to step over holes if needed.

Note 6.3 Some time in rare case the interpolation process miss some point, we can change the search algorithm through global variable `searchMethod`

```
searchMethod=0; // default value for fast search algorithm
searchMethod=1; // safe search algo, use brute force in case of missing
point // (warning can be very expensive in case of lot point of outside domain)
searchMethod=2; // use always the brute force very very expensive
```

Note 6.4 Step 3 requires an array of pointers such that each vertex points to one triangle of the triangulation.

Note 6.5 The operator `=` is the interpolation operator of `FreeFem++`, The continuous finite functions are extended by continuity to the outside of the domain. Try the following example

```

mesh Ths= square(10,10);
mesh Thg= square(30,30,[x*3-1,y*3-1]);
plot (Ths,Thg,ps="overlapTh.eps",wait=1);
fespace Ch(Ths,P2); fespace Dh(Ths,P2dc);
fespace Fh(Thg,P2dc);
Ch us= (x-0.5)*(y-0.5);
Dh vs= (x-0.5)*(y-0.5);
Fh ug=us,vg=vs;
plot (us,ug,wait=1,ps="us-ug.eps");           // see figure 6.12
plot (vs,vg,wait=1,ps="vs-vg.eps");           // see figure 6.13

```

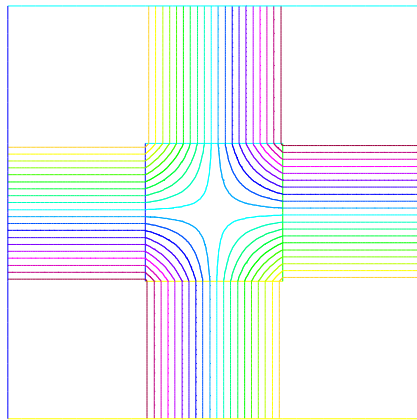


Figure 6.12: Extension of a continuous FE-function

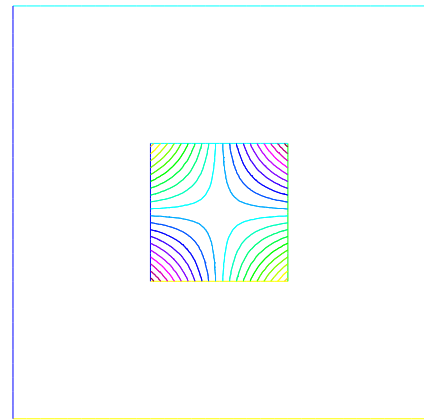


Figure 6.13: Extension of discontinuous FE-function, see warning 6

6.8 Keywords: Problem and Solve

For `FreeFem++` a problem must be given in variational form, so we need a bilinear form $a(u, v)$, a linear form $\ell(f, v)$, and possibly a boundary condition form must be added.

```

problem P(u,v) =
    a(u,v) - l(f,v)
    + (boundary condition);

```

Note 6.6 When you want to formulate the problem and to solve it in the same time, you can use the keyword `solve`.

6.8.1 Weak form and Boundary Condition

To present the principles of Variational Formulations or also called weak forms for the PDEs, let us take a model problem : a Poisson equation with Dirichlet and Robin Boundary condition .

The problem is: Find u a real function defined on domain Ω of \mathbb{R}^d ($d = 2, 3$) such that

$$-\nabla \cdot (\kappa \nabla u) = f, \quad \text{in } \Omega, \quad au + \kappa \frac{\partial u}{\partial n} = b \quad \text{on } \Gamma_r, \quad u = g \quad \text{on } \Gamma_d \quad (6.25)$$

where

- if $d = 2$ then $\nabla \cdot (\kappa \nabla u) = \partial_x(\kappa \partial_x u) + \partial_y(\kappa \partial_y u)$ with $\partial_x u = \frac{\partial u}{\partial x}$ and $\partial_y u = \frac{\partial u}{\partial y}$
- if $d = 3$ then $\nabla \cdot (\kappa \nabla u) = \partial_x(\kappa \partial_x u) + \partial_y(\kappa \partial_y u) + \partial_z(\kappa \partial_z u)$ with $\partial_x u = \frac{\partial u}{\partial x}$, $\partial_y u = \frac{\partial u}{\partial y}$ and $\partial_z u = \frac{\partial u}{\partial z}$
- the border $\Gamma = \partial\Omega$ is split in Γ_d and Γ_n such that $\Gamma_d \cap \Gamma_n = \emptyset$ and $\Gamma_d \cup \Gamma_n = \partial\Omega$,
- κ is a given positive function, such that $\exists \kappa_0 \in \mathbb{R}, \quad 0 < \kappa_0 \leq \kappa$.
- a a given non negative function,
- b a given function.

Note 6.7 This is the well known Neumann boundary condition if $a = 0$, and if Γ_d is empty. In this case the function appears in the problem just by its derivatives, so it is defined only up to a constant (if u is a solution then $u + c$ is also a solution).

Let v a regular test function null on Γ_d , by integration by parts we get

$$-\int_{\Omega} \nabla \cdot (\kappa \nabla u) v \, d\omega = \int_{\Omega} \kappa \nabla v \cdot \nabla u \, d\omega - \int_{\Gamma} v \kappa \frac{\partial u}{\partial \mathbf{n}} \, d\gamma = \int_{\Omega} f v \, d\omega \quad (6.26)$$

where if $d = 2$ the $\nabla v \cdot \nabla u = (\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y})$, where if $d = 3$ the $\nabla v \cdot \nabla u = (\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z})$, and where \mathbf{n} is the unitary outside normal of $\partial\Omega$.

Now we note that $\kappa \frac{\partial u}{\partial \mathbf{n}} = -au + g$ on Γ_r and $v = 0$ on Γ_d and $\partial\Omega = \Gamma_d \cup \Gamma_n$ thus

$$-\int_{\partial\Omega} v \kappa \frac{\partial u}{\partial \mathbf{n}} = \int_{\Gamma_r} auv - \int_{\Gamma_r} bv$$

The problem becomes:

Find $u \in V_g = \{v \in H^1(\Omega) / v = g \text{ on } \Gamma_d\}$ such that

$$\int_{\Omega} \kappa \nabla v \cdot \nabla u \, d\omega + \int_{\Gamma_r} auv \, d\gamma = \int_{\Omega} f v \, d\omega + \int_{\Gamma_r} bv \, d\gamma, \quad \forall v \in V_0 \quad (6.27)$$

where $V_0 = \{v \in H^1(\Omega) / v = 0 \text{ on } \Gamma_d\}$

Except in the case of Neumann conditions everywhere, the problem (6.27) is well posed when $\kappa \geq \kappa_0 > 0$.

Note 6.8 If we have only Neumann boundary condition, linear algebra tells us that the right hand side must be orthogonal to the kernel of the operator for the solution to exist. One way of writing the compatibility condition is:

$$\int_{\Omega} f \, d\omega + \int_{\Gamma} b \, d\gamma = 0$$

and a way to fix the constant is to solve for $u \in H^1(\Omega)$ such that:

$$\int_{\Omega} \varepsilon uv \, d\omega + \kappa \nabla v \cdot \nabla u \, d\omega = \int_{\Omega} f v \, d\omega + \int_{\Gamma_r} b v \, d\gamma, \quad \forall v \in H^1(\Omega) \quad (6.28)$$

where ε is a small parameter ($\sim 10^{-10}$).

Remark that if the solution is of order $\frac{1}{\varepsilon}$ then the compatibility condition is unsatisfied, otherwise we get the solution such that $\int_{\Omega} u = 0$, you can also add a Lagrange multiplier to solve the real mathematical problem like in the `examples++-tutorial/Laplace-lagrange-mult.edp` example.

In FreeFem++ , the bidimensional problem (6.27) becomes

```
problem Pw(u,v) =
  int2d(Th) ( kappa*( dx(u)*dx(u) + dy(u)*dy(u) ) ) //  $\int_{\Omega} \kappa \nabla v \cdot \nabla u \, d\omega$ 
+ int1d(Th,gn) ( a * u*v ) //  $\int_{\Gamma_r} a u v \, d\gamma$ 
- int2d(Th) ( f*v ) //  $\int_{\Omega} f v \, d\omega$ 
- int1d(Th,gn) ( b * v ) //  $\int_{\Gamma_r} b v \, d\gamma$ 
+ on(gd) (u= g) ; //  $u = g \text{ on } \Gamma_d$ 
```

where Th is a mesh of the the bidimensional domain Ω , and gd and gn are respectively the boundary label of boundary Γ_d and Γ_n .

And the the three dimensional problem (6.27) becomes

```
macro Grad(u) [dx(u),dy(u),dz(u)] // EOM : definition of the 3d Grad macro
problem Pw(u,v) =
  int3d(Th) ( kappa*( Grad(u)'*Grad(v) ) ) //  $\int_{\Omega} \kappa \nabla v \cdot \nabla u \, d\omega$ 
+ int2d(Th,gn) ( a * u*v ) //  $\int_{\Gamma_r} a u v \, d\gamma$ 
- int3d(Th) ( f*v ) //  $\int_{\Omega} f v \, d\omega$ 
- int2d(Th,gn) ( b * v ) //  $\int_{\Gamma_r} b v \, d\gamma$ 
+ on(gd) (u= g) ; //  $u = g \text{ on } \Gamma_d$ 
```

where Th is a mesh of the three dimensional domain Ω , and gd and gn are respectively the boundary label of boundary Γ_d and Γ_n .

6.9 Parameters affecting solve and problem

The parameters are FE functions real or complex, the number n of parameters is even ($n = 2 * k$), the k first function parameters are unknown, and the k last are test functions.

Note 6.9 If the functions are a part of vectoriel FE then you must give all the functions of the vectorial FE in the same order (see `laplaceMixte` problem for example).

Note 6.10 Don't mix complex and real parameters FE function.

Bug: 1 The mixing of fespace with different periodic boundary condition is not implemented. So all the finite element spaces used for test or unknown functions in a problem, must have the same type of periodic boundary condition or no periodic boundary condition. No clean message is given and the result is unpredictable, Sorry.

The parameters are:

solver= LU, CG, Crout,Cholesky,GMRES,sparsesolver, UMFPACK ...

The default solver is `sparsesolver` (it is equal to `UMFPACK` if not other sparse solver is defined) or is set to `LU` if no direct sparse solver is available. The storage mode of the matrix of the underlying linear system depends on the type of solver chosen; for `LU` the matrix is sky-line non symmetric, for `Crout` the matrix is sky-line symmetric, for `Cholesky` the matrix is sky-line symmetric positive definite, for `CG` the matrix is sparse symmetric positive, and for `GMRES`, `sparsesolver` or `UMFPACK` the matrix is just sparse.

eps= a real expression. ε sets the stopping test for the iterative methods like `CG`. Note that if ε is negative then the stopping test is:

$$||Ax - b|| < |\varepsilon|$$

if it is positive then the stopping test is

$$||Ax - b|| < \frac{|\varepsilon|}{||Ax_0 - b||}$$

init= boolean expression, if it is false or 0 the matrix is reconstructed. Note that if the mesh changes the matrix is reconstructed too.

precon= name of a function (for example `P`) to set the preconditioner. The prototype for the function `P` must be

```
func real[int] P(real[int] & xx) ;
```

tg= Huge value (10^{30}) used to implement Dirichlet boundary conditions, see page 180 for description.

tolpivot= set the tolerance of the pivot in `UMFPACK` (10^{-1}) and, `LU`, `Crout`, `Cholesky` factorisation (10^{-20}).

tolpivotsym= set the tolerance of the pivot sym in `UMFPACK`

strategy= set the integer `UMFPACK` strategy (0 by default).

6.10 Problem definition

Below v is the unknown function and w is the test function.

After the "=" sign, one may find sums of:

- identifier(s); this is the name given earlier to the variational form(s) (type `varf`) for possible reuse.

Remark, that the name in the "varf" of the unknown of test function is forgotten, we just used the order in argument list to recall name as in a C++ function, see note 6.15,

- the terms of the bilinear form itself: if K is a given function,
- bilinear part for 3d meshes `Th`

$$\text{int3d}(\text{Th}) (K * v * w) = \sum_{T \in \text{Th}} \int_T K v w$$

- $\text{int3d}(\text{Th}, 1) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w$
- $\text{int2d}(\text{Th}, 2, 5) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w$
- $\text{int2d}(\text{Th}, 1) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w$
- $\text{int2d}(\text{Th}, 2, 5) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w$
- $\text{int2d}(\text{Th}, \text{levelset}=\text{phi}) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{T, \phi=0} K v w$
- $\text{int2d}(\text{Th}, 1, \text{levelset}=\text{phi}) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi=0} K v w$
- $\text{intallfaces}(\text{Th}) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{\partial T} K v w$
- $\text{intallfaces}(\text{Th}, 1) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_{\partial T} K v w$
- they contribute to the sparse matrix of type `matrix` which, whether declared explicitly or not is constructed by `FreeFem++` .

- bilinear part for 2d meshes `Th`

- $\text{int2d}(\text{Th}) (K \star v \star w) = \sum_{T \in \text{Th}} \int_T K v w$
- $\text{int2d}(\text{Th}, 1) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w$
- $\text{int1d}(\text{Th}, 2, 5) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w$
- $\text{int1d}(\text{Th}, 1) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w$
- $\text{int1d}(\text{Th}, 2, 5) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w$
- $\text{int1d}(\text{Th}, \text{levelset}=\text{phi}) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{T, \phi=0} K v w$
- $\text{int1d}(\text{Th}, 1, \text{levelset}=\text{phi}) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi=0} K v w$
- $\text{intalledges}(\text{Th}) (K \star v \star w) = \sum_{T \in \text{Th}} \int_{\partial T} K v w$
- $\text{intalledges}(\text{Th}, 1) (K \star v \star w) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_{\partial T} K v w$
- they contribute to the sparse matrix of type `matrix` which, whether declared explicitly or not is constructed by `FreeFem++` .

- the right hand-side of the PDE in 3d, the terms of the linear form: for given functions K, f :

$$\begin{aligned}
- \text{int3d}(\text{Th}) (K * w) &= \sum_{T \in \text{Th}} \int_T K w \\
- \text{int3d}(\text{Th}, 1) (K * w) &= \sum_{T \in \text{Th}, T \in \Omega_l} \int_T K w \\
- \text{int2d}(\text{Th}, 2, 5) (K * w) &= \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K w \\
- \text{intallfaces}(\text{Th}) (f * w) &= \sum_{T \in \text{Th}} \int_{\partial T} f w \\
- &\text{a vector of type } \text{real}[\text{int}]
\end{aligned}$$

- the right hand-side of the PDE in 2d, the terms of the linear form: for given functions K, f :

$$\begin{aligned}
- \text{int2d}(\text{Th}) (K * w) &= \sum_{T \in \text{Th}} \int_T K w \\
- \text{int2d}(\text{Th}, 1) (K * w) &= \sum_{T \in \text{Th}, T \in \Omega_l} \int_T K w \\
- \text{int1d}(\text{Th}, 2, 5) (K * w) &= \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K w \\
- \text{intalledges}(\text{Th}) (f * w) &= \sum_{T \in \text{Th}} \int_{\partial T} f w \\
- &\text{a vector of type } \text{real}[\text{int}]
\end{aligned}$$

- The boundary condition terms :

- An "on" scalar form (for Dirichlet) : $\text{on}(1, u = g)$

The meaning is for all degree of freedom i of the boundary referred by "1", the diagonal term of the matrix $a_{ii} = \text{tgv}$ with the *terrible giant value* tgv ($=10^{30}$ by default) and the right hand side $b[i] = "(\Pi_h g)[i]" \times \text{tgv}$, where the $"(\Pi_h g)[i]"$ is the boundary node value given by the interpolation of g .

remark, if $\text{tgv} < 0$ then we put to 0 all term of the line i in the matrix, except diagonal term $a_{ii} = 1$, and $b[i] = "(\Pi_h g)[i]"$. (version i 3.10) .

- An "on" vectorial form (for Dirichlet) : $\text{on}(1, u1=g1, u2=g2)$ If you have vectorial finite element like RT0, the 2 components are coupled, and so you have : $b[i] = "(\Pi_h(g1, g2))[i]" \times \text{tgv}$, where Π_h is the vectorial finite element interpolant.
- a linear form on Γ (for Neumann in 2d) $-\text{int1d}(\text{Th}) (f * w)$ or $-\text{int1d}(\text{Th}, 3) (f * w)$
- a bilinear form on Γ or Γ_2 (for Robin in 2d) $\text{int1d}(\text{Th}) (K * v * w)$ or $\text{int1d}(\text{Th}, 2) (K * v * w)$.
- a linear form on Γ (for Neumann in 3d) $-\text{int2d}(\text{Th}) (f * w)$ or $-\text{int2d}(\text{Th}, 3) (f * w)$
- a bilinear form on Γ or Γ_2 (for Robin in 3d) $\text{int2d}(\text{Th}) (K * v * w)$ or $\text{int2d}(\text{Th}, 2) (K * v * w)$.

Note 6.11

- If needed, the different kind of terms in the sum can appear more than once.
- the integral mesh and the mesh associated to test function or unknown function can be different in the case of linear form.
- $N.x$, $N.y$ and $N.z$ are the normal's components.

Important: it is not possible to write in the same integral the linear part and the bilinear part such as in `int1d(Th) (K*v*w - f*w) .`

6.11 Numerical Integration

Let D be a N -dimensional bounded domain. For an arbitrary polynomial f of degree r , if we can find particular (quadrature) points ξ_j , $j = 1, \dots, J$ in D and (quadrature) constants ω_j such that

$$\int_D f(\mathbf{x}) = \sum_{\ell=1}^L c_\ell f(\xi_\ell) \quad (6.29)$$

then we have an error estimate (see Crouzeix-Mignot (1984)), and then there exists a constant $C > 0$ such that,

$$\left| \int_D f(\mathbf{x}) - \sum_{\ell=1}^L \omega_\ell f(\xi_\ell) \right| \leq C|D|h^{r+1} \quad (6.30)$$

for any function $r + 1$ times continuously differentiable f in D , where h is the diameter of D and $|D|$ its measure (a point in the segment $[q^i q^j]$ is given as

$$\{(x, y) | x = (1 - t)q_x^i + tq_x^j, y = (1 - t)q_y^i + tq_y^j, 0 \leq t \leq 1\}.$$

For a domain $\Omega_h = \sum_{k=1}^{n_t} T_k$, $\mathcal{T}_h = \{T_k\}$, we can calculate the integral over $\Gamma_h = \partial\Omega_h$ by

$$\begin{aligned} \int_{\Gamma_h} f(\mathbf{x}) ds &= \text{int1d}(\text{Th}) (f) \\ &= \text{int1d}(\text{Th}, \text{qfe}=\ast) (f) \\ &= \text{int1d}(\text{Th}, \text{qforder}=\ast) (f) \end{aligned}$$

where \ast stands for the name of the quadrature formula or the precision (order) of the Gauss formula.

Quadrature formula on an edge					
L	(qfe=)	qforder=	point in $[q^i q^j](=t)$	ω_ℓ	exact on $P_k, k =$
1	qf1pE	2	$1/2$	$ q^i q^j $	1
2	qf2pE	3	$(1 \pm \sqrt{1/3})/2$	$ q^i q^j /2$	3
3	qf3pE	6	$(1 \pm \sqrt{3/5})/2$ $1/2$	$(5/18) q^i q^j $ $(8/18) q^i q^j $	5
4	qf4pE	8	$(1 \pm \frac{\sqrt{525+70\sqrt{30}}}{35})/2$ $(1 \pm \frac{\sqrt{525-70\sqrt{30}}}{35})/2$	$\frac{18-\sqrt{30}}{72} q^i q^j $ $\frac{18+\sqrt{30}}{72} q^i q^j $	7
5	qf5pE	10	$(1 \pm \frac{\sqrt{245+14\sqrt{70}}}{21})/2$ $1/2$ $(1 \pm \frac{\sqrt{245-14\sqrt{70}}}{21})/2$	$\frac{322-13\sqrt{70}}{1800} q^i q^j $ $\frac{64}{225} q^i q^j $ $\frac{322+13\sqrt{70}}{1800} q^i q^j $	9
2	qf1pElump	2	0 +1	$ q^i q^j /2$ $ q^i q^j /2$	1

where $|q^i q^j|$ is the length of segment $\overline{q^i q^j}$. For a part Γ_1 of Γ_h with the label “1”, we can calculate the integral over Γ_1 by

$$\begin{aligned} \int_{\Gamma_1} f(x, y) ds &= \text{int1d}(\text{Th}, 1)(f) \\ &= \text{int1d}(\text{Th}, 1, \text{qfe=qf2pE})(f) \end{aligned}$$

The integrals over Γ_1, Γ_3 are given by

$$\int_{\Gamma_1 \cup \Gamma_3} f(x, y) ds = \text{int1d}(\text{Th}, 1, 3)(f)$$

For each triangle $T_k = [q^{k_1} q^{k_2} q^{k_3}]$, the point $P(x, y)$ in T_k is expressed by the *area coordinate* as $P(\xi, \eta)$:

$$\begin{aligned} |T_k| &= \frac{1}{2} \begin{vmatrix} 1 & q_x^{k_1} & q_y^{k_1} \\ 1 & q_x^{k_2} & q_y^{k_2} \\ 1 & q_x^{k_3} & q_y^{k_3} \end{vmatrix} & D_1 &= \begin{vmatrix} 1 & x & y \\ 1 & q_x^{k_2} & q_y^{k_2} \\ 1 & q_x^{k_3} & q_y^{k_3} \end{vmatrix} & D_2 &= \begin{vmatrix} 1 & q_x^{k_1} & q_y^{k_1} \\ 1 & x & y \\ 1 & q_x^{k_3} & q_y^{k_3} \end{vmatrix} & D_3 &= \begin{vmatrix} 1 & q_x^{k_1} & q_y^{k_1} \\ 1 & q_x^{k_2} & q_y^{k_2} \\ 1 & x & y \end{vmatrix} \\ \xi &= \frac{1}{2} D_1 / |T_k| & \eta &= \frac{1}{2} D_2 / |T_k| & \text{then } 1 - \xi - \eta &= \frac{1}{2} D_3 / |T_k| \end{aligned}$$

For a two dimensional domain or a border of three dimensional domain $\Omega_h = \sum_{k=1}^{n_t} T_k$, $\mathcal{T}_h = \{T_k\}$, we can calculate the integral over Ω_h by

$$\begin{aligned} \int_{\Omega_h} f(x, y) &= \text{int2d}(\text{Th})(f) \\ &= \text{int2d}(\text{Th}, \text{qft}=\ast)(f) \\ &= \text{int2d}(\text{Th}, \text{qforder}=\ast)(f) \end{aligned}$$

where \ast stands for the name of quadrature formula or the order of the Gauss formula.

Quadrature formula on a triangle					
L	qft=	qforder=	point in T_k	ω_ℓ	exact on $P_k, k =$
1	qf1pT	2	$(\frac{1}{3}, \frac{1}{3})$	$ T_k $	1
3	qf2pT	3	$(\frac{1}{2}, \frac{1}{2})$ $(\frac{1}{2}, 0)$ $(0, \frac{1}{2})$	$ T_k /3$ $ T_k /3$ $ T_k /3$	2
7	qf5pT	6	$(\frac{1}{3}, \frac{1}{3})$ $(\frac{6-\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21})$ $(\frac{6-\sqrt{15}}{21}, \frac{9+2\sqrt{15}}{21})$ $(\frac{9+2\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21})$ $(\frac{6+\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21})$ $(\frac{6+\sqrt{15}}{21}, \frac{9-2\sqrt{15}}{21})$ $(\frac{9-2\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21})$	$0.225 T_k $ $\frac{(155-\sqrt{15}) T_k }{1200}$ $\frac{(155-\sqrt{15}) T_k }{1200}$ $\frac{(155-\sqrt{15}) T_k }{1200}$ $\frac{(155+\sqrt{15}) T_k }{1200}$ $\frac{(155+\sqrt{15}) T_k }{1200}$ $\frac{(155+\sqrt{15}) T_k }{1200}$	5
3	qf1pTlump		$(0, 0)$ $(1, 0)$ $(0, 1)$	$ T_k /3$ $ T_k /3$ $ T_k /3$	1
9	qf2pT4P1		$(\frac{1}{4}, \frac{3}{4})$ $(\frac{3}{4}, \frac{1}{4})$ $(0, \frac{1}{4})$ $(0, \frac{3}{4})$ $(\frac{1}{4}, 0)$ $(\frac{3}{4}, 0)$ $(\frac{1}{4}, \frac{1}{4})$ $(\frac{1}{4}, \frac{1}{2})$ $(\frac{1}{2}, \frac{1}{4})$	$ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /6$ $ T_k /6$ $ T_k /6$	1
15	qf7pT	8	see [38] for detail		7
21	qf9pT	10	see [38] for detail		9

For a three dimensional domain $\Omega_h = \sum_{k=1}^{n_t} T_k$, $\mathcal{T}_h = \{T_k\}$, we can calculate the integral over Ω_h by

$$\begin{aligned}
 \int_{\Omega_h} f(x, y) &= \text{int3d}(\mathcal{T}_h)(f) \\
 &= \text{int3d}(\mathcal{T}_h, \text{qfV}=\ast)(f) \\
 &= \text{int3d}(\mathcal{T}_h, \text{qforder}=\ast)(f)
 \end{aligned}$$

where \ast stands for the name of quadrature formula or the order of the Gauss formula.

Quadrature formula on a tetrahedron					
L	qfV=	qforder=	point in $T_k \in \mathbb{R}^3$	ω_ℓ	exact on $P_k, k =$
1	qfV1	2	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$ T_k $	1
4	qfV2	3	$G4(0.58\dots, 0.13\dots, 0.13\dots)$	$ T_k /4$	2
14	qfV5	6	$G4(0.72\dots, 0.092\dots, 0.092\dots)$ $G4(0.067\dots, 0.31\dots, 0.31\dots)$ $G6(0.45\dots, 0.045\dots, 0.45\dots)$	$0.073\dots T_k $ $0.11\dots T_k $ $0.042\dots T_k $	5
4	qfV1lump		$G4(1, 0, 0)$	$ T_k /4$	1

Where $G4(a, b, b)$ such that $a + 3b = 1$ is the set of the four point in barycentric coordinate

$$\{(a, b, b, b), (b, a, b, b), (b, b, a, b), (b, b, b, a)\}$$

and where $G6(a, b, b)$ such that $2a + 2b = 1$ is the set of the six points in barycentric coordinate

$$\{(a, a, b, b), (a, b, a, b), (a, b, b, a), (b, b, a, a), (b, a, b, a), (b, a, a, b)\}.$$

Note 6.12 These tetrahedral quadrature formulae come from <http://www.cs.kuleuven.be/~nines/research/ecf/mtables.html>

Note 6.13 By default, we use the formula which is exact for polynomials of degree 5 on triangles or edges (in bold in three tables).

This possible to add an own quadrature formulae with using plugin "qf11to25" on segment, triangle or Tetrahedron. The quadrature formulae in D dimension is a bidimensional array of size $N_q \times (D + 1)$ such that the $D + 1$ value of on row $i = 0, \dots, N_p - 1$ are $w^i, \hat{x}_1^i, \dots, \hat{x}_D^i$ where w^i is the weight of the quadrature point, and $1 - \sum_{k=1}^D \hat{x}_k^i, \hat{x}_1^i, \dots, \hat{x}_D^i$ is the barycentric coordinate the quadrature point.

```

//      just for test ... (version 3.19-1)
load "qf11to25"                                     //      load plugin

//      Quadrature on segment
real[int,int] qq1=[
    [0.5,0],
    [0.5,1]];

QF1 qf1(1,qq1);                                     //      def of quadrature formulae qf1 on segment
//      remark:
//      1 is the order of the quadrature exact for polynome of degree < 1)

//      Quadrature on triangle
real[int,int] qq2=[
    [1./3,0,0],
    [1./3.,1,0],
    [1./3.,0,1]];

QF2 qf2(1,qq2);                                     //      def of quadrature formulae qf2 on triangle
//      remark:
//      1 is the order of the quadrature exact for polynome of degree < 1)
//      so must have  $\Rightarrow \sum w^i = 1$ 

//      Quadrature on Tetrahedron
real[int,int] qq3=[
    [1./4,0,0,0],
    [1./4.,1,0,0],
    [1./4.,0,1,0],
    [1./4.,0,0,1]];

QF3 qf3(1,qq3);                                     //      def of quadrature formulae qf3 on get.
//      remark:
//      1 is the order of the quadrature exact for polynome of degree < 1)

```

```

// verification in 1d end 2d..
real I1 = int1d(Th,qfe=qf1)(x^2) ;
real I1l = int1d(Th,qfe=qf1pElump)(x^2) ;

real I2 = int2d(Th,qft=qf2)(x^2) ;
real I2l = int2d(Th,qft=qf1pTlump)(x^2) ;

cout << I1 << " == " << I1l << endl;
cout << I2 << " == " << I2l << endl;
assert( abs(I1-I1l) < 1e-10);
assert( abs(I2-I2l) < 1e-10);

```

the output is

```

1.67 == 1.67
0.335 == 0.335

```

6.12 Variational Form, Sparse Matrix, PDE Data Vector

In FreeFem++ it is possible to define variational forms, and use them to build matrices and vectors and store them to speed-up the script (4 times faster here).

For example let us solve the Thermal Conduction problem of section 3.4.

The variational formulation is in $L^2(0, T; H^1(\Omega))$; we shall seek u^n satisfying

$$\forall w \in V_0; \quad \int_{\Omega} \frac{u^n - u^{n-1}}{\delta t} w + \kappa \nabla u^n \nabla w + \int_{\Gamma} \alpha (u^n - u_{ue}) w = 0$$

where $V_0 = \{w \in H^1(\Omega) / w|_{\Gamma_{24}} = 0\}$.

So the to code the method with the matrices $A = (A_{ij})$, $M = (M_{ij})$, and the vectors $u^n, b^n, b', b'', b_{cl}$ (notation if w is a vector then w_i is a component of the vector).

$$u^n = A^{-1} b^n, \quad b' = b_0 + M u^{n-1}, \quad b'' = \frac{1}{\varepsilon} b_{cl}, \quad b_i^n = \begin{cases} b''_i & \text{if } i \in \Gamma_{24} \\ b'_i & \text{else if } i \notin \Gamma_{24} \end{cases} \quad (6.31)$$

Where with $\frac{1}{\varepsilon} = \text{tgv} = 10^{30}$:

$$A_{ij} = \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{ and } j = i \\ \int_{\Omega} w_j w_i / dt + k(\nabla w_j \cdot \nabla w_i) + \int_{\Gamma_{13}} \alpha w_j w_i & \text{else if } i \notin \Gamma_{24}, \text{ or } j \neq i \end{cases} \quad (6.32)$$

$$M_{ij} = \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{ and } j = i \\ \int_{\Omega} w_j w_i / dt & \text{else if } i \notin \Gamma_{24}, \text{ or } j \neq i \end{cases} \quad (6.33)$$

$$b_{0,i} = \int_{\Gamma_{13}} \alpha u_{ue} w_i \quad (6.34)$$

$$b_{cl} = u^0 \quad \text{the initial data} \quad (6.35)$$

```

// file thermal-fast.edp in examples++-tutorial

func fu0 =10+90*x/6;

```

```
func k = 1.8*(y<0.5)+0.2;
real ue = 25. , alpha=0.25, T=5, dt=0.1 ;
```

```
mesh Th=square(30,5,[6*x,y]);
fespace Vh(Th,P1);
```

```
Vh u0=fu0,u=u0;
```

Create three variational formulation, and build the matrices A, M .

```
varf vthermic (u,v)= int2d(Th) (u*v/dt + k*(dx(u) * dx(v) + dy(u) * dy(v)))
+ int1d(Th,1,3) (alpha*u*v) + on(2,4,u=1);
```

```
varf vthermic0(u,v) = int1d(Th,1,3) (alpha*ue*v);
```

```
varf vMass (u,v)= int2d(Th) ( u*v/dt) + on(2,4,u=1);
```

```
real tgv = 1e30;
matrix A= vthermic(Vh,Vh,tgv=tgv,solver=CG);
matrix M= vMass(Vh,Vh);
```

Now, to build the right hand size we need 4 vectors.

```
real[int] b0 = vthermic0(0,Vh); // constant part of the RHS
real[int] bcn = vthermic(0,Vh); // tgv on Dirichlet boundary node ( !=0 )
// we have for the node i : i ∈ Γ24 ⇔ bcn[i] ≠ 0
real[int] bcl=tgv*u0[]; // the Dirichlet boundary condition part
```

Note 6.14 The boundary condition is implemented by penalization and vector bcn contains the contribution of the boundary condition $u=1$, so to change the boundary condition, we have just to multiply the vector $bc[]$ by the current value f of the new boundary condition term by term with the operator $.*$. Section 9.6.2 Examples++-tutorial/StokesUzawa.edp gives a real example of using all this features.

And the new version of the algorithm is now:

```
ofstream ff("thermic.dat");
for(real t=0;t<T;t+=dt){
  real[int] b = b0 ; // for the RHS
  b += M*u[]; // add the the time dependent part
  // lock boundary part:
  b = bcn ? bcl : b ; // do ∀i: b[i] = bcn[i] ? bcl[i] : b[i] ;
  u[] = A^-1*b;
  ff << t << " "<<u(3,0.5)<<endl;
  plot(u);
}
for(int i=0;i<20;i++){
  cout<<dy(u)(6.0*i/20.0,0.9)<<endl;
  plot(u,fill=true,wait=1,ps="thermic.eps");
}
```

Note 6.15 *The functions appearing in the variational form are formal and local to the `varf` definition, the only important thing is the order in the parameter list, like in*

```
varf vb1 ([u1,u2],q) = int2d(Th) ( (dy(u1)+dy(u2)) *q) + int2d(Th) (1*q);
varf vb2 ([v1,v2],p) = int2d(Th) ( (dy(v1)+dy(v2)) *p) + int2d(Th) (1*p);
```

To build matrix A from the bilinear part the variational form a of type **varf** simply write:

```
A = a(Vh,Wh [, ...] );
//   where
//   Vh is "fespace" for the unknow fields with a correct number of component
//   Wh is "fespace" for the test fields with a correct number of component
```

Possible named parameters in " [, ...] " are

solver= LU, CG, Crout, Cholesky, GMRES, sparsesolver, UMFPACK ...

The default solver is GMRES.

The storage mode of the matrix of the underlying linear system depends on the type of solver chosen; for LU the matrix is sky-line non symmetric, for Crout the matrix is sky-line symmetric, for Cholesky the matrix is sky-line symmetric positive definite, for CG the matrix is sparse symmetric positive, and for GMRES, sparsesolver or UMFPACK the matrix is just sparse.

factorize = if true then do the matrix factorization for LU, Cholesky or Crout, the default value is *false*.

eps= a real expression. ε sets the stopping test for the iterative methods like CG. Note that if ε is negative then the stopping test is:

$$\|Ax - b\| < |\varepsilon|$$

if it is positive then the stopping test is

$$\|Ax - b\| < \frac{|\varepsilon|}{\|Ax_0 - b\|}$$

precon= name of a function (for example P) to set the preconditioner. The prototype for the function P must be

```
func real[int] P(real[int] & xx) ;
```

tg= Huge value (10^{30}) used to implement Dirichlet boundary conditions.

tolpivot= set the tolerance of the pivot in UMFPACK (10^{-1}) and, LU, Crout, Cholesky factorisation (10^{-20}).

tolpivotsym= set the tolerance of the pivot sym in UMFPACK

strategy= set the integer UMFPACK strategy (0 by default).

Note 6.16 *The line of the matrix corresponding to the space Wh and the column of the matrix corresponding to the space Vh .*

To build the dual vector b (of type `real[int]`) from the linear part of the variational form a do simply

```
real b(Vh.ndof);
b = a(0,Vh);
```

A first example to compute the area of each triangle K of mesh Th , just do:

```
fespace Nh(Th,P0); // the space function constant / triangle
Nh areaK;
varf varea(unused,chiK) = int2d(Th)(chiK);
etaK[] = varea(0,Ph);
```

Effectively, the basic functions of space Nh , are the characteristic function of the element of Th , and the numbering is the numeration of the element, so by construction:

$$\text{etaK}[i] = \int 1_{|K_i} = \int_{K_i} 1;$$

Now, we can use this to compute error indicators like in examples `AdaptResidualErrorIndicator.edp` in directory `examples++-tutorial`.

First to compute a continuous approximation to the function h "density mesh size" of the mesh Th .

```
fespace Vh(Th,P1);
Vh h;
real[int] count(Th.nv);
varf vmeshsizen(u,v)=intalldges(Th,qfnpE=1)(v);
varf vedgecount(u,v)=intalldges(Th,qfnpE=1)(v/lenEdge);
// computation of the mesh size
// -----
count=vedgecount(0,Vh); // number of edge / vertex
h[]=vmeshsizen(0,Vh); // sum length edge / vertex
h[]=h[]./count; // mean lenght edge / vertex
```

To compute error indicator for Poisson equation :

$$\eta_K = \int_K h_K^2 |(f + \Delta u_h)|^2 + \int_{\partial K} h_e \left| \frac{\partial u_h}{\partial n} \right|^2$$

where h_K is size of the longest edge (`hTriangle`), h_e is the size of the current edge (`lenEdge`), n the normal.

```
fespace Nh(Th,P0); // the space function contant / triangle
Nh etak;
varf vetaK(unused,chiK) =
    intalldges(Th)(chiK*lenEdge*square(jump(N.x*dx(u)+N.y*dy(u))))
    +int2d(Th)(chiK*square(hTriangle*(f+dxx(u)+dyy(u)))) ;
etak[] = vetaK(0,Ph);
```

We add automatic expression optimization by default, if this optimization creates problems, it can be removed with the keyword `optimize` as in the following example :

```
varf a(u1,u2)= int2d(Th,optimize=false)( dx(u1)*dx(u2) + dy(u1)*dy(u2) )
+ on(1,2,4,u1=0) + on(3,u1=1) ;
```


Remark, it is all possible to build interpolation matrix, like in the following example:

```

mesh TH = square(3,4);
mesh th = square(2,3);
mesh Th = square(4,4);

fespace VH(TH,P1);
fespace Vh(th,P1);
fespace Wh(Th,P1);

matrix B= interpolate(VH,Vh);           //    build interpolation matrix Vh->VH
matrix BB= interpolate(Wh,Vh);         //    build interpolation matrix Vh->Wh

```

and after some operations on sparse matrices are available for example

```

int N=10;
real [int,int] A(N,N);           //    a full matrix
real [int] a(N),b(N);
A =0;
for (int i=0;i<N;i++)
{
    A(i,i)=1+i;
    if (i+1 < N)    A(i,i+1)=-i;
    a[i]=i;
}
b=A*b;
cout << "xxxx\n";
matrix sparseA=A;
cout << sparseA << endl;
sparseA = 2*sparseA+sparseA';
sparseA = 4*sparseA+sparseA*5;
matrix sparseB=sparseA+sparseA+sparseA; ;
cout << "sparseB = " << sparseB(0,0) << endl;

```

6.13 Interpolation matrix

It is also possible to store the matrix of a linear interpolation operator from a finite element space V_h to another W_h to `interpolate(W_h, V_h, \dots)` a function. Note that the continuous finite functions are extended by continuity outside of the domain.

The named parameters of function `interpolate` are:

inside= set true to create zero-extension.

t= set true to get the transposed matrix

op= set an integer written below

- 0 the default value and interpolate of the function
- 1 interpolate the ∂_x
- 2 interpolate the ∂_y
- 3 interpolate the ∂_z

U2Vc= set the which is the component of W_h come in V_h in interpolate process in a int array so the size of the array is number of component of W_h , if the put -1 then component is set to 0, like in the following example: (by default the component number is unchanged).

```

fespace V4h(Th4, [P1,P1,P1,P1]);
fespace V3h(Th, [P1,P1,P1]);
int[int] u2vc=[1,3,-1]; // -1 => put zero on the component
matrix IV34= interpolate(V3h,V4h,inside=0,U2Vc=u2vc); // V3h <- V4h
V4h [a1,a2,a3,a4]=[1,2,3,4];
V3h [b1,b2,b3]=[10,20,30];
b1[]=IV34*a1[];

```

So here we have: $b1 == 2, b2 == 4, b3 == 0 \dots$

Example 6.2 (mat.interpol.edp)

```

mesh Th=square(4,4);
mesh Th4=square(2,2,[x*0.5,y*0.5]);
plot(Th,Th4,ps="ThTh4.eps",wait=1);
fespace Vh(Th,P1); fespace Vh4(Th4,P1);
fespace Wh(Th,P0); fespace Wh4(Th4,P0);

matrix IV= interpolate(Vh,Vh4); // here the function is
// extended by continuity

cout << " IV Vh<-Vh4 " << IV << endl;
Vh v, vv; Vh4 v4=x*y;
v=v4; vv[]= IV*v4[]; // here v == vv =>
real[int] diff= vv[] - v[];
cout << " || v - vv || = " << diff.linfty << endl;
assert( diff.linfty<= 1e-6);
matrix IV0= interpolate(Vh,Vh4,inside=1); // here the function is
// extended by zero

cout << " IV Vh<-Vh4 (inside=1) " << IV0 << endl;
matrix IVt0= interpolate(Vh,Vh4,inside=1,t=1);
cout << " IV Vh<-Vh4^t (inside=1) " << IVt0 << endl;
matrix IV4t0= interpolate(Vh4,Vh);
cout << " IV Vh4<-Vh^t " << IV4t0 << endl;
matrix IW4= interpolate(Wh4,Wh);
cout << " IV Wh4<-Wh " << IW4 << endl;
matrix IW4V= interpolate(Wh4,Vh);
cout << " IV Wh4<-Vh " << IW4 << endl;

```

Build interpolation matrix A at a array of points $(xx[j], yy[j]), i = 0, 2$ here

$$a_{ij} = dop(w_c^i(xx[j], yy[j]))$$

where w_i is the basic finite element function, c the component number, dop the type of diff operator like in op def.

```

real[int] xx=[.3,.4],yy=[.1,.4];
int c=0,dop=0;
matrix Ixx= interpolate(Vh,xx,yy,op=dop,composante=c);
cout << Ixx << endl;
Vh ww;
real[int] dd=[1,2];
ww[]= Ixx*dd;

```

6.14 Finite elements connectivity

Here, we show how to get informations on a finite element space $W_h(\mathcal{T}_n, *)$, where “*” may be P1, P2, P1nc, etc.

- `Wh.nt` gives the number of element of W_h
- `Wh.ndof` gives the number of degrees of freedom or unknown
- `Wh.ndofK` gives the number of degrees of freedom on one element
- `Wh(k, i)` gives the number of i th degrees of freedom of element k .

See the following example:

Example 6.3 (FE.edp) `mesh Th=square(5,5);`
`fespace Wh(Th,P2);`
`cout << " nb of degree of freedom : " << Wh.ndof << endl;`
`cout << " nb of degree of freedom / ELEMENT : " << Wh.ndofK << endl;`
`int k= 2, kdf= Wh.ndofK ;;` // element 2
`cout << " df of element " << k << ":" ;`
`for (int i=0;i<kdf;i++) cout << Wh(k,i) << " ";`
`cout << endl;`

The output is:

```
Nb Of Nodes = 121
Nb of DF = 121
FESpace:Gibbs: old skyline = 5841 new skyline = 1377
nb of degree of freedom : 121
nb of degree of freedom / ELEMENT : 6
df of element 2:78 95 83 87 79 92
```


Chapter 7

Visualization

Results created by the finite element method can be a huge set of data, so it is very important to render them easy to grasp. There are two ways of visualization in FreeFem++ : One, the default view, supports the drawing of meshes, isovalues of real FE-functions and of vector fields, all by the command **plot** (see Section 7.1 below). For publishing purpose, FreeFem++ can store these plots as postscript files.

Another method is to use external tools, for example, gnuplot (see Section 7.2), medit (see Section 7.3) using the command **system** to launch them and/or to save the data in text files.

7.1 Plot

With the command plot, meshes, isovalues of scalar functions and vector fields can be displayed. The parameters of the plot command can be , meshes, real FE functions , arrays of 2 real FE functions, arrays of two arrays of double, to plot respectively a mesh, a function, a vector field, or a curve defined by the two arrays of double.

Note 7.1 *The length of an arrow is always bound to be in [5%,5%] of the screen size, to see something (else it will only look like porcupine).*

The parameters are

wait= boolean expression to wait or not (by default no wait). If true we wait for a keyboard event or mouse event, they respond to an event by the following characters

enter try to show plot

p previous plot (10 plots saved)

? show this help

+, - zoom in/out around the cursor 3/2 times

= reset vue

r refresh plot

up, down, left, right special keys to translate

3 switch 3d/2d plot keys :

z, Z) focal zoom unzoom

H, h switch increase or decrease the Z scale of the plot

mouse motion

- **left button** rotate
- **right button** zoom (ctrl+button on mac)
- **right button +alt** translate (alt+ctrl+button on mac)

a, A increase or decrease the arrow size

B switch between show border meshes or not

i, I update or not: the min/max bound of the functions to the window

n, N decrease or increase the number of iso value array

b switch between black and white or color plotting

g switch between grey or color plotting

f switch between filling iso or iso line

l switch between lighting or not

v switch between show or not the numerical value of colors

m switch between show or not meshes

w window dump in file ffglutXXXX.ppm

***** keep/unkeep viewpoint for next plot

k complex data / change view type

ESC close the graphics process before version 3.22, after no way to close.

otherwise do nothing.

ps= string expression for the name of the file to save the plot in postscript (sorry no save of 3d plot)

coef= the vector arrow size between arrow unit and domain unit.

fill= fill color between iso-values (mandatory of P0 finite element).

cmm= string expression to write the graphic window into

value= to plot the value of isolines and the value of vector arrows.

aspectratio= boolean to be sure that the aspect ratio of plot is preserved or not.

bb= array of 2 array (like `[[0.1, 0.2], [0.5, 0.6]]`), to set the bounding box and specify a partial view where the box defined by the two corner points `[0.1, 0.2]` and `[0.5, 0.6]`.

nbiso= (int) sets the number of isovalues (20 by default)

nbarrow= (int) sets the number of colors of arrow values (20 by default)

viso= sets the array of isovalues (an array `real[int]` of increasing values)

varrow= sets the array of color arrows values (an array `real[int]`)

bw= (bool) sets or not the plot in black and white color.

grey= (bool) sets or not the plot in grey color.

hsv= (array of float) to defined color of 3*n value in HSV color model declared for example by

```
real[int] colors = [h1,s1,v1,... , hn,vn,vn];
```

where h_i, s_i, v_i is the i th color to defined the color table.

boundary= (bool) to plot or not the boundary of the domain (true by default).

dim= (int) sets dim of the plot 2d or 3d (2 by default)

add= Not used

prev= set the default graphic state to the previous state

ech= <d> Not used

ZScale= <d> Not used

WhiteBackground= Not used

OpaqueBorders= Not used

BorderAsMesh= Not used

ShowMeshes= Not used

ColorScheme= <l> Not used

ArrowShape= <l> Not used

ArrowSize= <d> Not used

ComplexDisplay= <l> Not used

LabelColors= Not used

ShowAxes= Not used

CutPlane= Not used

CameraPosition= Not used

CameraFocalPoint= Not used

CameraViewUp= Not used

CameraViewAngle= <d> Not used

CameraClippingRange= Not used

CutPlaneOrigin= Not used

CutPlaneNormal= Not used

WindowIndex= set glut window for display for multi windows graphics.

For example:

```

real[int]  xx(10),yy(10);
mesh Th=square(5,5);
fespace Vh(Th,P1);
Vh uh=x*x+y*y,vh=-y^2+x^2;
int i;

                                                                    //    compute a cut
for (i=0;i<10;i++)
{
    x=i/10.; y=i/10.;
    xx[i]=i;
    yy[i]=uh;
                                //    value of uh at point (i/10. , i/10.)
}
plot(Th,uh,[uh,vh],value=true,ps="three.eps",wait=true);           //    figure 7.1
//    zoom on box defined by the two corner points [0.1,0.2] and [0.5,0.6]
plot(uh,[uh,vh],bb=[[0.1,0.2],[0.5,0.6]],
    wait=true,greyscale=1,fill=1,value=1,ps="threeg.eps");           //    figure 7.2
plot([xx,yy],ps="likegnu.eps",wait=true);                           //    figure 7.3

```

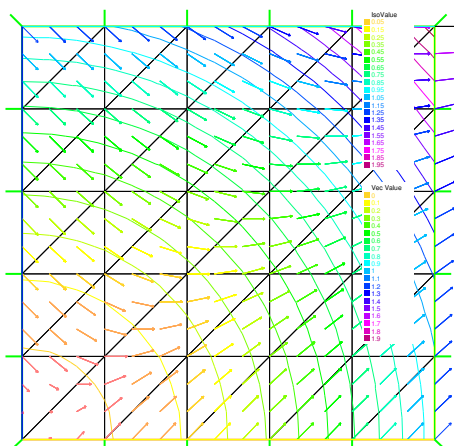


Figure 7.1: mesh, isovalue, and vector

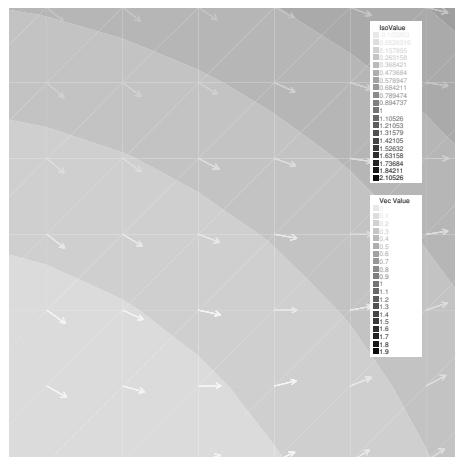


Figure 7.2: enlargement in grey of isovalue, and vector

To change the color table and to choose the value of iso line you can do :

```

//    from: http://en.wikipedia.org/wiki/HSV_color_space
//    The HSV (Hue, Saturation, Value) model,
//    defines a color space in terms of three constituent components:
//
//    HSV color space as a color wheel 7.4
//    Hue, the color type (such as red, blue, or yellow):
//    Ranges from 0-360 (but normalized to 0-100% in some applications Here)
//    Saturation, the "vibrancy" of the color: Ranges from 0-100%
//    The lower the saturation of a color, the more "grayness" is present
//    and the more faded the color will appear.
//    Value, the brightness of the color:
//    Ranges from 0-100%

```

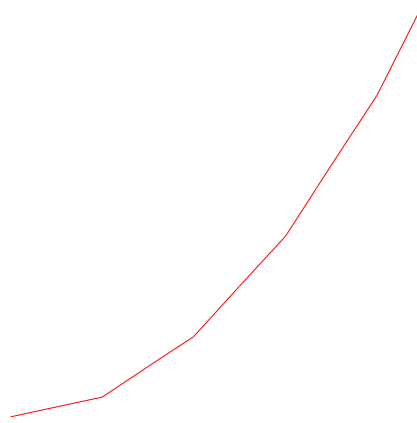



Figure 7.3: Plots a cut of uh . Note that a refinement of the same can be obtained in combination with gnuplot

```
//
real[int] colorhsv=[
    4./6., 1 , 0.5,
    4./6., 1 , 1,
    5./6., 1 , 1,
    1, 1. , 1,
    1, 0.5 , 1
];
real[int] viso(31);

for (int i=0;i<viso.n;i++)
    viso[i]=i*0.1;

plot (uh,viso=viso(0:viso.n-1),value=1,fill=1,wait=1,hsv=colorhsv);
```

// color hsv model
 // dark blue
 // blue
 // magenta
 // red
 // light red

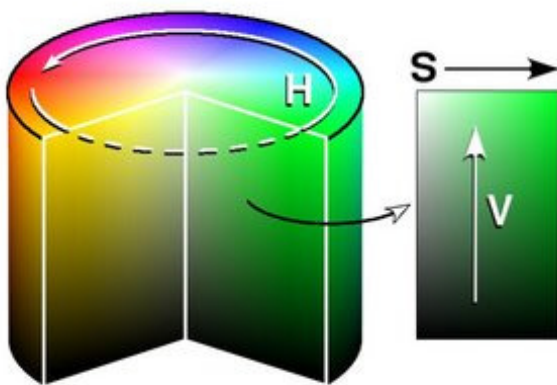


Figure 7.4: hsv color cylinder

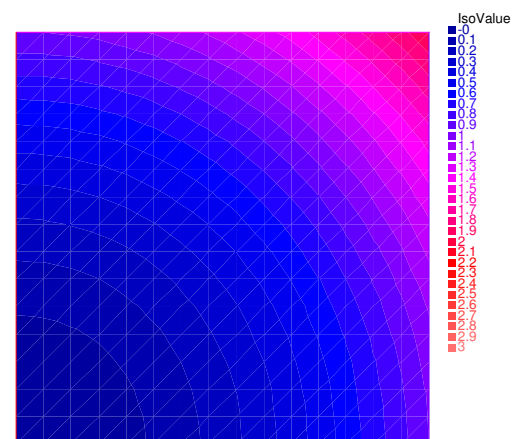


Figure 7.5: isovalue with an other color table

7.2 link with gnuplot

Example 3.2 shows how to generate a gnu-plot from a FreeFem++ file. Let us present here another technique which has the advantage of being online, i.e. one doesn't need to quit FreeFem++ to generate a gnu-plot. But this work only if gnuplot¹ is installed, and only on unix computer. Add to the previous example:

```
{
    //      file for gnuplot
    ofstream gnu("plot.gp");
    for (int i=0;i<=n;i++)
    {
        gnu << xx[i] << " " << yy[i] << endl;
    }
} //      the file plot.gp is close because the variable gnu is delete

//      to call gnuplot command and wait 5 second (thanks to unix command)
//      and make postscript plot
exec("echo 'plot \"plot.gp\" w l \
pause 5 \
set term postscript \
set output \"gnuplot.eps\" \
replot \
quit' | gnuplot");
```

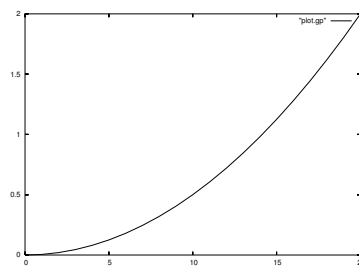


Figure 7.6: Plots a cut of uh with gnuplot

7.3 link with medit

As said above, medit² is a freeware display package by Pascal Frey using OpenGL. Then you may run the following example.

Remark: Now medit software is include in FreeFem++ under ffmedit name.
Now with version 3.2 or better

```
load "medit"
mesh Th=square(10,10,[2*x-1,2*y-1]);
fespace Vh(Th,P1);
Vh u=2-x*x-y*y;
medit ("mm",Th,u);
```

¹<http://www.gnuplot.info/>

²<http://www-rocq.inria.fr/gamma/medit/medit.html>

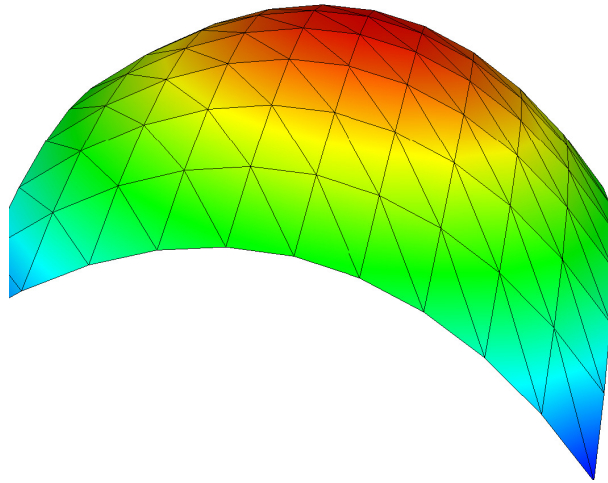


Figure 7.7: medit plot

Before:

```

mesh Th=square(10,10,[2*x-1,2*y-1]);
fespace Vh(Th,P1);
Vh u=2-x*x-y*y;
savemesh(Th,"mm",[x,y,u*.5]);           // save mm.points and mm.faces file
                                           // for medit
                                           // build a mm.bb file

{
  ofstream file("mm.bb");
  file << "2 1 1 " << u[].n << " 2 \n";
  for (int j=0;j<u[].n ; j++)
    file << u[][j] << endl;
}

                                           // call medit command
exec("ffmedit mm");

                                           // clean files on unix OS
exec("rm mm.bb      mm.faces  mm.points");

```


Chapter 8

Algorithms and Optimization

The complete example is in `algo.edp` file.

8.1 conjugate Gradient/GMRES

Suppose we want to solve the Euler problem (here x has nothing to do with the reserved variable for the first coordinate in `FreeFem++`): find $x \in \mathbb{R}^n$ such that

$$\nabla J(x) = \left(\frac{\partial J}{\partial x_i}(x) \right) = 0 \quad (8.1)$$

where J is a functional (to minimize for example) from \mathbb{R}^n to \mathbb{R} .

If the function is convex we can use the conjugate gradient to solve the problem, and we just need the function (named `dJ` for example) which compute ∇J , so the parameters are the name of that function with prototype

```
func real[int] dJ(real[int] & xx);
```

which compute ∇J , and a vector `x` of type (of course the number 20 can be changed)

```
real[int] x(20);
```

to initialize the process and get the result.

Given an initial value $\mathbf{x}^{(0)}$, a maximum number i_{\max} of iterations, and an error tolerance $0 < \epsilon < 1$: Put $\mathbf{x} = \mathbf{x}^{(0)}$ and write

```
NLCG( $\nabla J$ , x, precon= $M$ , nbiter= $i_{\max}$ , eps= $\epsilon$ , stop=stopfunc);
```

will give the solution of \mathbf{x} of $\nabla J(\mathbf{x}) = 0$. We can omit parameters `precon`, `nbiter`, `eps`, `stop`. Here M is the preconditioner whose default is the identity matrix. The stopping test is

$$\|\nabla J(\mathbf{x})\|_P \leq \epsilon \|\nabla J(\mathbf{x}^{(0)})\|_P$$

Writing the minus value in `eps=`, i.e.,

```
NLCG( $\nabla J$ , x, precon= $M$ , nbiter= $i_{\max}$ , eps= $-\epsilon$ );
```

we can use the stopping test

$$\|\nabla J(\mathbf{x})\|_P^2 \leq \epsilon$$

The parameters of these three functions are:

nbiter= set the number of iteration (by default 100)

precon= set the preconditioner function (P for example) by default it is the identity, remark the prototype is `func real[int] P(real[int] &x)`.

eps= set the value of the stop test ε ($= 10^{-6}$ by default) if positive then relative test $\|\nabla J(x)\|_P \leq \varepsilon \|\nabla J(x_0)\|_P$, otherwise the absolute test is $\|\nabla J(x)\|_P^2 \leq |\varepsilon|$.

veps= set and return the value of the stop test, if positive then relative test $\|\nabla J(x)\|_P \leq \varepsilon \|\nabla J(x_0)\|_P$, otherwise the absolute test is $\|\nabla J(x)\|_P^2 \leq |\varepsilon|$. The return value is minus the real stop test (remark: it is useful in loop).

stop= *stopfunc* add your test function to stop before (after version 3.31). The prototype for the function *stopfunc* is

```
func bool stopfunc(int iter, real[int] u, real[int] g)
```

where u is the current solution, and g the current gradient not preconditioned.

Example 8.1 (algo.edp) For a given function b , let us find the minimizer u of the functional

$$J(u) = \frac{1}{2} \int_{\Omega} f(|\nabla u|^2) - \int_{\Omega} ub$$

$$f(x) = ax + x - \ln(1+x), \quad f'(x) = a + \frac{x}{1+x}, \quad f''(x) = \frac{1}{(1+x)^2}$$

under the boundary condition $u = 0$ on $\partial\Omega$.

```
func real J(real[int] & u)
{
  Vh w; w[] = u; // copy array u in the finite element function w
  real r = int2d(Th) (0.5*f( dx(w)*dx(w) + dy(w)*dy(w) ) - b*w) ;
  cout << "J(u) =" << r << " " << u.min << " " << u.max << endl;
  return r;
}

// -----

Vh u=0; // the current value of the solution
Ph alpha; // of store df(|\nabla u|^2)
int iter=0;
alpha=df( dx(u)*dx(u) + dy(u)*dy(u) ); // optimization

func real[int] dJ(real[int] & u)
{
  int verb=verbosity; verbosity=0;
  Vh w; w[] = u; // copy array u in the finite element function w
  alpha=df( dx(w)*dx(w) + dy(w)*dy(w) ); // optimization
  varf au(uh,vh) = int2d(Th) ( alpha*( dx(w)*dx(vh) + dy(w)*dy(vh) ) - b*vh)
    + on(1,2,3,4,u=0);
  u= au(0,Vh);
  verbosity=verb;
  return u; // warning no return of local array
}
```

We want to construct also a preconditioner C with solving the problem: find $u_h \in V_{0h}$ such that

$$\forall v_h \in V_{0h}, \quad \int_{\Omega} \alpha \nabla u_h \cdot \nabla v_h = \int_{\Omega} b v_h$$

where $\alpha = f'(|\nabla u|^2)$. */

```

varf alap(uh,vh)= int2d(Th) ( alpha * ( dx(uh)*dx(vh) + dy(uh)*dy(vh) ) )
                  + on(1,2,3,4,uh=0);

varf amass(uh)= int2d(Th) ( uh*vh) + on(1,2,3,4,uh=0);

matrix Amass = alap(Vh,Vh,solver=CG); //
matrix Alap= alap(Vh,Vh,solver=Cholesky,factorize=1); //

// the preconditionner function
func real[int] C(real[int] & u)
{
    real[int] w = Amass*u;
    u = Alap^-1*w;
    return u; // no return of local array variable
}

/* To solve the problem, we make 10 iteration of the conjugate gradient, recompute the preconditioner and restart the conjugate gradient: */

verbosity=5;
int conv=0;
real eps=1e-6;
for(int i=0;i<20;i++)
{
    conv=NLCG(dJ,u[],nbiter=10,precon=C,veps=eps); //
    if (conv) break; // if converge break loop
    alpha=df( dx(u)*dx(u) + dy(u)*dy(u) ); // recompute alpha optimization
    Alap = alap(Vh,Vh,solver=Cholesky,factorize=1);
    cout << " restart with new preconditionner " << conv
          << " eps =" << eps << endl;
}

plot (u,wait=1,cmm="solution with NLCG");

```

For a given symmetric positive matrix A , consider the quadratic form

$$J(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T A \mathbf{x} - \mathbf{b}^T \mathbf{x}$$

then $J(\mathbf{x})$ is minimized by the solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$. In this case, we can use the function `LinearCG`

```
AffineCG(A, x, precon=M, nbiter=i_max, eps=±ε, stop=stp);
```

If A is not symmetric, we can use GMRES(Generalized Minimum Residual) algorithm by

```
AffineGMRES(A, x, precon=M, nbiter=i_max, eps=±ε);
```

Also, we can use the non-linear version of GMRES algorithm (the functional J is just convex)

```
AffineGMRES(∇J, x, precon=M, nbiter=i_max, eps=±ε);
```

For detail of these algorithms, refer to [14][Chapter IV, 1.3].

8.2 Algorithms for Unconstrained Optimization

Two algorithms of COOOL a package [27] are interfaced with the Newton Raphson method (call Newton) and the BFGS method. These two ones are directly available in FreeFem (no dynamical link to load). Be careful with these algorithms, because their implementation uses full matrices. We also provide several optimization algorithms from the NLOpt library [42] as well as an interface for Hansen's implementation of CMAES (a MPI version of this one is also available). These last algorithms can be found as dynamical links in the `example++-load` folder as the `ff-NLOpt` and `CMA-ES` files (`CMA-ES-MPI` from the `example++-mpi` folder for the mpi version).

8.2.1 Example of utilization for BFGS or CMAES

```

real [int] b(10),u(10);
func real J(real [int] & u)
{
    real s=0;
    for (int i=0;i<u.n;i++)
        s +=(i+1)*u[i]*u[i]*0.5 - b[i]*u[i];
    cout << "J ="<< s << " u =" << u[0] << " " << u[1] << "...\\n" ;
    return s;
}

// the grad of J (this is a affine version (the RHS is in )
func real [int] DJ(real [int] &u)
{
    for (int i=0;i<u.n;i++)
        u[i]=(i+1)*u[i]-b[i];
    return u;
};

// return of global variable ok

b=1; u=2; // set right hand side and initial gest
BFGS (J,dJ,u,eps=1.e-6,nbiter=20,nbiterline=20);
cout << "BFGS: J(u) = " << J(u) << endl;

```

Using the CMA evolution strategy is almost the same, except that, as it is a derivative free optimizer, the `dJ` argument is omitted and there are some other named parameters to control the behaviour of the algorithm. With the same objective function as above, an example of utilization would be (see `cmaes-VarIneq.edp` for a complete example):

```

load "ff-cmaes"
... // define J, u and all here
real min = cmaes(J,u,stopTolFun=1e-6,stopMaxIter=3000);
cout << "minimal value is " << min << " for u = " << u << endl;

```

This algorithm works with a normal multivariate distribution in the parameters space and try to adapt its covariance matrix using the information provides by the successive function evaluations (see [43] for more details). Thus, some specific parameters can be passed to control the starting distribution, size of the sample generations etc... Named parameters for this are the following :

seed= Seed for random number generator (`val` is an integer). No specified value will lead to a clock based seed initialization.

initialStdDev= Value for the standard deviations of the initial covariance matrix (`val` is a real). If the value σ is passed, the initial covariance matrix will be set to σI . The expected initial distance between initial X and the *argmin* should be roughly `initialStdDev`. Default is 0.3.

initialStdDevs= Same as above except that the argument is an array allowing to set a value of the initial standard deviation for each parameter. Entries differing by several orders of magnitude should be avoided (if it can't be, try rescaling the problem).

stopTolFun= Stops the algorithm if function values differences are smaller than the passed one, default is 10^{-12} .

stopTolFunHist= Stops the algorithm if function value differences of the best values are smaller than the passed one, default is 0 (unused).

stopTolX= Stopping criteria triggered if step sizes in the parameters space are smaller than this real value, default is 0.

stopTolXFactor= Stopping criteria triggered when the standard deviation increases more than this value. The default value is 10^3 .

stopMaxFunEval= Stops the algorithm when `stopMaxFunEval` function evaluations have been done. Set to $900(n+3)^2$ by default, where n is the parameters space dimension.

stopMaxIter= Integer stopping the search when `stopMaxIter` generations has been sampled. Unused by default.

popsiz= Integer value used to change the sample size. The default value is $4 + \lfloor 3 \ln(n) \rfloor$, see [43] for more details. Increasing the population size usually improves the global search capabilities at the cost of an at most linear reduction of the convergence speed with respect to `popsiz`.

paramFile= This `string` type parameter allows the user to pass all the parameters using an external file as in Hansen's original code. More parameters related to the CMA-ES algorithm can be changed with this file. A sample of it can be found in the `examples++-load/ffcMAES/` folder under the name `initials.par`. Note that the parameters passed to the CMAES function in the FreeFem script will be ignored if an input parameters file is given.

8.3 IPOPT

The `ff-Ipopt` package is an interface for the IPOPT [44] optimizer. IPOPT is a software library for large scale, non-linear, constrained optimization. Detailed informations about it are in [44] and <https://projects.coin-or.org/Ipopt>. It implements a primal-dual interior point method along with filter method based line searches. IPOPT need a direct sparse symmetric linear solver. If your version of FreeFem has been compiled with the `--enable-downlad` tag, it will automatically be linked with a sequential version of MUMPS. An alternative to MUMPS would be to download the HSL subroutines (see <http://www.coin-or.org/Ipopt/documentation/node16.html>) and place them in the `/ipopt/Ipopt-3.10.2/ThirdParty/HSL` directory of the FreeFem++ downloads folder before compiling.

8.3.1 Short description of the algorithm

In this section, we give a very brief glimpse at the underlying mathematics of IPOPT. For a deeper introduction on interior methods for nonlinear smooth optimization, one may consults [45], or [44] for more IPOPT specific elements. IPOPT is designed to perform optimization for both equality and inequality constrained problems. Though, nonlinear inequalities are rearranged before the beginning of the optimization process in order to restrict the panel of nonlinear constraints to those of the equality kind. Each nonlinear inequality ones are transformed into a pair of simple bound inequality and nonlinear equality constraint by the introduction of as many slack variables as is needed : $c_i(x) \leq 0$ becomes $c_i(x) + s_i = 0$ and $s_i \leq 0$, where s_i is added to the initial variables of the problems x_i . Thus, for convenience, we will assume that the minimization problem does not contain any nonlinear inequality constraint. It means that, given a function $f : \mathbb{R}^n \mapsto \mathbb{R}$, we want to find :

$$\begin{aligned} x_0 = \underset{x \in V}{\operatorname{argmin}} f(x) \\ \text{with } V = \{x \in \mathbb{R}^n \mid c(x) = 0 \text{ and } x_l \leq x \leq x_u\} \end{aligned} \quad (8.2)$$

Where $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $x_l, x_u \in \mathbb{R}^n$ and inequalities hold componentwise. The f function as well as the constraints c should be twice-continuously differentiable.

As a barrier method, interior points algorithms try to find a Karush-Kuhn-Tucker point for (8.2) by solving a sequence of problems, unconstrained with respect to the inequality constraints, of the form :

$$\text{for a given } \mu > 0, \text{ find } x_\mu = \underset{x \in \mathbb{R}^n \mid c(x)=0}{\operatorname{argmin}} B(x, \mu) \quad (8.3)$$

Where μ is a positive real number and $B(x, \mu) = f(x) - \mu \sum_{i=1}^n \ln(x_{u,i} - x_i) - \mu \sum_{i=1}^m \ln(x_i - x_{l,i})$.

The remaining equality constraints are handled with the usual Lagrange multipliers method. If the sequence of barrier parameters μ converge to 0, intuition suggests that the sequence of minimizers of (8.3) converge to a local constrained minimizer of (8.2). For a given μ , (8.3) is solved by finding $(x_\mu, \lambda_\mu) \in \mathbb{R}^n \times \mathbb{R}^m$ such that :

$$\nabla B(x_\mu, \mu) + \sum_{i=1}^m \lambda_{\mu,i} \nabla c_i(x_\mu) = \nabla B(x_\mu, \mu) + J_c(x_\mu)^T \lambda_\mu = 0 \quad \text{and} \quad c(x_\mu) = 0 \quad (8.4)$$

The derivations for ∇B only holds for the x variables, so that :

$$\nabla B(x, \mu) = \nabla f(x) + \begin{pmatrix} \mu/(x_{u,1} - x_1) \\ \vdots \\ \mu/(x_{u,n} - x_n) \end{pmatrix} - \begin{pmatrix} \mu/(x_1 - x_{l,1}) \\ \vdots \\ \mu/(x_n - x_{l,n}) \end{pmatrix}$$

If we respectively call $z_u(x, \mu) = (\mu/(x_{u,1} - x_1), \dots, \mu/(x_{u,n} - x_n))$ and $z_l(x, \mu)$ the other vector appearing in the above equation, then the optimum (x_μ, λ_μ) satisfies :

$$\nabla f(x_\mu) + J_c(x_\mu)^T \lambda_\mu + z_u(x_\mu, \mu) - z_l(x_\mu, \mu) = 0 \quad \text{and} \quad c(x_\mu) = 0 \quad (8.5)$$

In this equation, the z_l and z_u vectors seems to play the role of Lagrange multipliers for the simple bounds inequalities, and indeed, when $\mu \rightarrow 0$, they converge toward some suitable Lagrange multipliers for the KKT conditions, provided some technical assumptions are fulfilled (see [45]).

Equation 8.5 is solved by performing a Newton method in order to find a solution of (8.4) for each of the decreasing values of μ . Some order 2 conditions are also taken into account to avoid convergence to local maximizer, see [45] for precision about them. In the most classical IP algorithms, the Newton method is directly applied to (8.4). This is in most case inefficient due to frequent computation of infeasible points. These difficulties are avoided in Primal-Dual interior points methods

where (8.4) is transformed into an extended system where z_u and z_l are treated as unknowns and the barrier problems are finding $(x, \lambda, z_u, z_l) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$ such that :

$$\begin{cases} \nabla f(x) + J_c(x)^T \lambda + z_u - z_l &= 0 \\ c(x) &= 0 \\ (X_u - X)z_u - \mu e &= 0 \\ (X - X_l)z_l - \mu e &= 0 \end{cases} \quad (8.6)$$

Where if a is a vector of \mathbb{R}^n , A denotes the diagonal matrix $A = (a_i \delta_{ij})_{1 \leq i, j \leq n}$ and $e \in \mathbb{R}^n = (1, 1, \dots, 1)$. Solving this nonlinear system by the Newton methods is known as being the *primal-dual* interior points method. Here again, more details are available in [45]. Most actual implementations introduce features in order to globalize the convergence capability of the method, essentially by adding some line-search steps to the Newton algorithm, or by using trust regions. For the purpose of IPOPT, this is achieved by a *filter line search* methods, the details of which can be found in [?].

More IPOPT specific features or implementation details can be found in [44]. We will just retain that IPOPT is a smart Newton method for solving constrained optimization problem, with global convergence capabilities due to a robust line search method (in the sense that the algorithm will converge no matter the initializer). Due to the underlying Newton method, the optimization process requires expressions of all derivatives up to the order 2 of the fitness function as well as those of the constraints. For problems whose hessian matrices are difficult to compute or lead to high dimensional dense matrices, it is possible to use a BFGS approximation of these objects at the cost of a much slower convergence rate.

8.3.2 IPOPT in FreeFem++

Calling the IPOPT optimizer in a FreeFem++ script is done with the IPOPT function included in the `ff-Ipopt` dynamic library. IPOPT is designed to solve constrained minimization problem in the form :

$$\begin{aligned} \text{find } x_0 &= \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} f(x) \\ \text{s.t. } \begin{cases} \forall i \leq n, x_i^{\text{lb}} \leq x_i \leq x_i^{\text{ub}} & \text{(simple bounds)} \\ \forall i \leq m, c_i^{\text{lb}} \leq c_i(x) \leq c_i^{\text{ub}} & \text{(constraints functions)} \end{cases} \end{aligned} \quad (8.7)$$

Where `ub` and `lb` stand for "upper bound" and "lower bound". If for some $i, 1 \leq i \leq m$ we have $c_i^{\text{lb}} = c_i^{\text{ub}}$, it means that c_i is an equality constraint, and an inequality one if $c_i^{\text{lb}} < c_i^{\text{ub}}$.

There are different ways to pass the fitness function and constraints. The more general one is to define the functions using the keyword `func`. Any returned matrix must be a sparse one (type `matrix`, not a `real[int, int]`) :

```
func real J(real[int] &X) {...} // Fitness Function, returns a scalar
func real[int] gradJ(real[int] &X) {...} // Gradient is a vector

func real[int] C(real[int] &X) {...} // Constraints
func matrix jacC(real[int] &X) {...} // Constraints jacobian
```

Warning 1 : in the current version of FreeFem++ , returning a matrix object local to a function block leads to undefined results. For each sparse matrix returning function you define, an extern matrix object has to be declared, whose associated function will overwrite and return on each call. Here is an example for `jacC` :

```
matrix jacCBuffer; // just declare, no need to define yet
```

```

func matrix jacC(real[int] &X)
{
    ...
    return jacCBuffer;
}

```

// fill jacCBuffer

Warning 2: IPOPT requires the structure of each matrix at the initialization of the algorithm. Some errors may occur if the matrices are not constant and are built with the matrix $A = [I, J, C]$ syntax, or with an intermediary full matrix (`real[int, int]`), because any null coefficient is discarded during the construction of the sparse matrix. It is also the case when making matrices linear combinations, for which any zero coefficient will result in the suppression of the matrix from the combination. Some controls are available to avoid such problems. Check the named parameters descriptions (`checkindex`, `structhess` and `structjac` can help). We strongly advice to use `varf` as much as possible for the matrix forging.

The hessian returning function is somewhat different because it has to be the hessian of the lagrangian function : $(x, \sigma_f, \lambda) \mapsto \sigma_f \nabla^2 f(x) + \sum_{i=1}^m \lambda_i \nabla^2 c_i(x)$ where $\lambda \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}$. Your hessian function should then have the following prototype :

```

matrix hessianLBuffer;
func matrix hessianL(real[int] &X, real sigma, real[int] &lambda) {...}

```

// just to keep it in mind

If the constraints functions are all affine, or if there are only simple bounds constraints or no constraint at all, the lagrangian hessian is equal to the fitness function hessian, one can then omit the sigma and lambda parameters :

```

matrix hessianJBuffer;
func matrix hessianJ(real[int] &X) {...}

```

// Hessian prototype when constraints are affine

When these functions are defined, IPOPT is called this way :

```

real[int] Xi = ... ;
IPOPT(J, gradJ, hessianL, C, jacC, Xi, /*some named parameters*/ );

```

// starting point

If the hessian is omitted, the interface will tell IPOPT to use the (L)BFGS approximation (it can also be enabled with a named parameter, see further). Simple bounds or unconstrained problems do not require the constraints part, so the following expressions are valid :

```

IPOPT(J, gradJ, C, jacC, Xi, ... );
IPOPT(J, gradJ, hessianJ, Xi, ... );
IPOPT(J, gradJ, Xi, ... );

```

// IPOPT with BFGS
// Newton IPOPT without constraints
// BFGS, no constraints

Simple bounds are passed using the `lb` and `ub` named parameters, while constraints bounds are passed with the `clb` and `cub` ones. Unboundedness in some directions can be achieved by using the $1e^{19}$ and $-1e^{19}$ values that IPOPT recognizes as $+\infty$ and $-\infty$:

```

real[int] xlb(n), xub(n), clb(m), cub(m);
...
IPOPT(J, gradJ, hessianL, C, jacC, Xi, lb=xlb, ub=xub, clb=clb, cub=cub, /*some other
named parameters*/ );

```

// fill the arrays...

P2 fitness function and affine constraints function : In the case where the fitness function or constraints function can be expressed respectively in the following forms :

$$\begin{aligned} \forall x \in \mathbb{R}^n, f(x) &= \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle & (A, b) &\in \mathcal{M}_{n,n}(\mathbb{R}) \times \mathbb{R}^n \\ \text{or, } C(x) &= Ax + b & (A, b) &\in \mathcal{M}_{n,m}(\mathbb{R}) \times \mathbb{R}^m \end{aligned}$$

where A and b are constant, it is possible to directly pass the (A, b) pair instead of defining 3 (or 2) functions. It also indicates to IPOPT that some objects are constant and that they have to be evaluated only once, thus avoiding multiple copies of the same matrix. The syntax is :

```

// Affine constraints with "standard" fitness function
matrix A= ... ; // Linear part of the constraints
real[int] b = ... ; // Constant part of constraints
IPOPT(J,gradJ,hessianJ, [A,b] ,Xi, /*bounds and named params*/);
// [b,A] would work as well... Scatterbrains pampering...
```

Note that if you define the constraints in this way, they doesn't contribute to the hessian, so the hessian should only take one `real[int]` as argument.

```

// Affine constraints and P2 fitness func:
matrix A= ... ; // Bilinear form matrix
real[int] b = ... ; // Linear contribution to f
matrix Ac= ... ; // Linear part of the constraints
real[int] bc= ... ; // Constant part of constraints
IPOPT([A,b], [Ac,bc] ,Xi, /*bounds and named params*/);
```

If both objective and constraints functions are given this way, it automatically activates the IPOPT `mehrotra_algorithm` option (better for linear and quadratic programming according to the documentation). Otherwise, this option can only be set through the option file (see the named parameters section).

A spurious case is the one of defining f in this manner while using standard functions for the constraints :

```

matrix A= ... ; // Bilinear form matrix
real[int] b = ... ; // Linear contribution to f
func real[int] C(real[int] &X) {...} // Constraints
func matrix jacC(real[int] &X) {...} // Constraints jacobian
IPOPT([A,b],C,jacC,Xi, /*bounds and named params*/);
```

Indeed, when passing $[A, b]$ in order to define f , the lagrangian hessian is automatically build has the constant $x \mapsto A$ function, with no way to add possible constraints contributions, leading to incorrect second order derivatives. So, a problem should be defined like that in only two cases : 1) constraints are nonlinear but you want to use the BFGS mode (then add `bfgs=1` to the named parameter), 2) constraints are affine, but in this case, why not passing them in the same way? Here are some other valid definitions of the problem (cases when f is a pure quadratic or linear form, or C a pure linear function, etc...) :

```

// Pure quadratic f - A is a matrix:
IPOPT(A, /*constraints args*/, Xi, /*bounds and named params*/);
// Pure linear f - b i a real[int] :
IPOPT(b, /*constraints args*/, Xi, /*bounds and named params*/);
// linear constraints - Ac is a matrix
IPOPT(/*fitness func args*/, Ac, Xi, /*bounds and named params*/);
```

Returned Value : The IPOPT function returns an error code of type `int`. A zero value is obtained when the algorithm succeeds and positive values reflects the fact that IPOPT encounters minor troubles. Negative values reveals more problematic cases. The associated IPOPT return tags are listed in the table below. The IPOPT pdf documentation provides more accurate description of these return status :

Success		Failures	
0	Solve_Succeeded	-1	Maximum_Iterations_Exceeded
1	Solved_To_Acceptable_Level	-2	Restoration_Failed
2	Infeasible_Problem_Detected	-3	Error_In_Step_Computation
3	Search_Direction_Becomes_Too_Small	-4	Maximum_CpuTime_Exceeded
4	Diverging_Iterates		
5	User_Requested_Stop		
6	Feasible_Point_Found		
Problem definition issues		Critical errors	
-10	Not_Enough_Degrees_Of_Freedom	-100	Unrecoverable_Exception
-11	Invalid_Problem_Definition	-101	NonIpopt_Exception_Thrown
-12	Invalid_Option	-102	Insufficient_Memory
-13	Invalid_Number_Detected	-199	Internal_Error

Named Parameters : The available named parameters in this interface are those we thought to be the most subject to variations from one optimization to another, plus a few ones that are interface specific. Though, as one could see at <http://www.coin-or.org/Ipopt/documentation/node59.html>, there are many parameters that can be changed within IPOPT, affecting the algorithm behaviour. These parameters can still be controlled by placing an option file in the execution directory. Note that IPOPT's pdf documentation may provides more informations than the previously mentioned online version for certain parameters. The in-script available parameters are :

lb, ub : `real[int]` for lower and upper simple bounds upon the search variables, must be of size n (search space dimension). If two components of same index in these arrays are equal then the corresponding search variable is fixed. By default IPOPT will remove any fixed variable from the optimization process and always use the fixed value when calling functions. It can be changed using the `fixedvar` parameter.

clb, cub : `real[int]` of size m (number of constraints) for lower and upper constraints bounds. Equality between two components of same index i in `clb` and `cub` reflect an equality constraint.

structjacc : To pass the greatest possible structure (indexes of non null coefficients) of the constraints jacobian under the form `[I, J]` where `I` and `J` are two integer arrays. If not defined, the structure of the constraints jacobian, evaluated in `Xi`, is used (no issue if the jacobian is constant or always defined with the same `varf`, hazardous if it is with triplet array or if a full matrix is involved).

structhess : Same as above but for the hessian function (unused if f is P2 or less and constraints are affine). Here again, keep in mind that it is the hessian of the lagrangian function (which is equal to the hessian of f only if constraints are affine). If no structure is given with this parameter, the lagrangian hessian is evaluated on the starting point, with $\sigma = 1$ and $\lambda = (1, 1, \dots, 1)$ (it is safe if all the constraints and fitness function Hessians are constant or build with `varf`, and here again less reliable if built with triplet array or full matrix).

- checkindex** : A `bool` that triggers an index dichotomic search when matrices are copied from FreeFem functions to IPOPT arrays. It is used to avoid wrong index matching when some null coefficients are removed from the matrices by FreeFem. It will not solve the problems arising when a too small structure has been given at the initialization of the algorithm. Enabled by default (except in cases where all matrices are obviously constant).
- warmstart** : If set to `true`, the constraints dual variables λ , and simple bounds dual variables are initialized with the values of the arrays passed to `lm`, `lz` and `uz` named parameters (see below).
- lm** : `real[int]` of size m , which is used to get the final values of the constraints dual variables λ and/or initialize them in case of a warm start (the passed array is also updated to the last dual variables values at the end of the algorithm).
- lz, uz** : `real[int]` of size n to get the final values and/or initialize (in case of warm start) the dual variables associated to simple bounds.
- tol** : `real`, convergence tolerance for the algorithm, the default value is 10^{-8} .
- maxiter** : `int`, maximum number of iterations with 3000 as default value.
- maxcputime** : `real` value, maximum runtime duration. Default is 10^6 (almost 11 days and a half).
- bfgs** : `bool` enabling or not the (low-storage) BFGS approximation of the lagrangian hessian. It is set to `false` by default, unless there is no way to compute the hessian with the functions that have been passed to IPOPT.
- derivativetest** : Used to perform a comparison of the derivatives given to IPOPT with finite differences computation. The possible string values are : `"none"` (default), `"first-order"`, `"second-order"` and `"only-second-order"`. The associated derivative error tolerance can be changed via the option file. One should not care about any error given by it before having tried, and failed, to perform a first optimization.
- dth** : Perturbation parameter for the derivative test computations with finite differences. Set by default to 10^{-8} .
- dttol** : Tolerance value for the derivative test error detection (default value unknown yet, maybe 10^{-5}).
- optfile** : `string` parameter to specify the IPOPT option file name. IPOPT will look for a `ipopt.opt` file by default. Options set in the file will overwrite those defined in the FreeFem script.
- printlevel** : An `int` to control IPOPT output print level, set to 5 by default, the possible values are from 0 to 12. A description of the output informations is available in the pdf documentation of IPOPT.
- fixedvar** : `string` for the definition of simple bounds equality constraints treatment : use `"make_parameter"` (default value) to simply remove them from the optimization process (the functions will always be evaluated with the fixed value for those variables), `"make_constraint"` to treat them as any other constraint or `"relax_bounds"` to relax fixing bound constraints.
- mustategy** : a `string` to choose the update strategy for the barrier parameter μ . The two possible tags are `"monotone"`, to use the monotone (Fiacco-McCormick) strategy, or `"adaptive"` (default setting).

muinit : real positive value for the barrier parameter initialization. It is only relevant when `mustrategy` has been set to `monotone`.

pivtol : real value to set the pivot tolerance for the linear solver. A smaller number pivots for sparsity, a larger number pivots for stability. The value has to be in the $[0, 1]$ interval and is set to 10^{-6} by default.

brf : Bounds relax factor : before starting the optimization, the bounds given by the user are relaxed. This option sets the factor for this relaxation. If it is set to zero, then the bounds relaxation is disabled. This `real` has to be positive and its default value is 10^{-8} .

objvalue : An identifier to a real type variable to get the last value of the objective function (best value in case of succes).

mumin : Minimal value for the barrier parameter μ , a `real` with 10^{-11} as default value.

linesearch : A boolean which disables the line search when set to `false`. The line search is activated by default. When disabled, the method becomes a standard Newton algorithm other the primal-dual system. The global convergence is then no longer assured, meaning that many initializers could lead to diverging iterates. But on the other hand, it can be useful when trying to catch a precise local minimum without having some out of control process making the iterate caught by some other near optimum.

8.4 Some short examples using IPOPT

Example 8.2 (IpoptVI.edp) *A very simple example consisting in, given two functions f and g (defined on $\Omega \subset \mathbb{R}^2$), minimizing $J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} fu$, with $u \leq g$ almost everywhere :*

```
load "ff-Ipopt"; // load the interface
int nn=20; // mesh quality
mesh Th=square(nn,nn); // build a square mesh
fespace Vh(Th,P1); // finite element space

func f = 1.; // rhs function
real r=0.03,s=0.1; // some parameters for g
func g = r - r/2*exp(-0.5*(square(x-0.5)+square(y-0.5))/square(s));
// g is constant minus a gaussian

macro Grad(u) [dx(u),dy(u)] // the gradient operator
varf vP(u,v) = int2d(Th)(Grad(u)'*Grad(v)) - int2d(Th)(f*v);
```

Here we build the matrix and second member associated to the functional to minimize once and for all. The `[A,b]` syntax for the fitness function is then used to pass it to IPOPT.

```
matrix A = vP(Vh,Vh,solver=CG);
real[int] b = vP(0,Vh);
```

We use simple bounds to impose the boundary condition $u = 0$ on $\partial\Omega$, as well as the $u \leq g$ condition.

```
Vh lb=-1.e19; // lower-unbounded in the interior
Vh ub=g; // upper-bounded by g in the interior
varf vGamma(u,v) = on(1,2,3,4,u=1);
```



```

real[int] onGamma=vGamma(0,Vh);
ub[] = onGamma ? 0. : ub[];           // enforcing the boundary condition
lb[] = onGamma ? 0. : lb[];

Vh u=0;                               // starting point
IPOPT([A,b],u[],lb=lb[],ub=ub[]);  // solve the problem
plot(u,wait=1);

```

Example 8.3 (IpoptVI2.edp) Let Ω be a domain of \mathbb{R}^2 , $f_1, f_2 \in L^2(\Omega)$ and $g_1, g_2 \in L^2(\partial\Omega)$ four given functions with $g_1 \leq g_2$ almost everywhere. We define the space :

$$V = \{(v_1, v_2) \in H^1(\Omega)^2; v_1|_{\partial\Omega} = g_1, v_2|_{\partial\Omega} = g_2, v_1 \leq v_2 \text{ a.e.}\}$$

as well as the functional $J : H^1(\Omega)^2 \rightarrow \mathbb{R}$:

$$J(v_1, v_2) = \frac{1}{2} \int_{\Omega} |\nabla v_1|^2 - \int_{\Omega} f_1 v_1 + \frac{1}{2} \int_{\Omega} |\nabla v_2|^2 - \int_{\Omega} f_2 v_2$$

The problem consists in finding (numerically) two functions $(u_1, u_2) = \underset{(v_1, v_2) \in V}{\operatorname{argmin}} J(v_1, v_2)$.

```

load "ff-IpOpt";

mesh Th=square(10,10);
fespace Vh(Th, [P1,P1] );
fespace Wh(Th, [P1] );
int iter=0;

func f1 = 10;           // right hand sides
func f2 = -15;
func g1 = -0.1;          // Boundary conditions functions
func g2 = 0.1;

while(++iter)           // mesh adaptation loop
{
  macro Grad(u) [dx(u),dy(u)] // gradient macro
  varf vP([u1,u2],[v1,v2]) = int2d(Th) (Grad(u1)'*Grad(v1) + Grad(u2)'*Grad(v2))
  - int2d(Th) (f1*v1+f2*v2);

  matrix A = vP(Vh,Vh); // Fitness function matrix...
  real[int] b = vP(0,Vh); // and linear form

  int[int] II1=[0],II2=[1]; // Constraints matrix
  matrix C1 = interpolate (Wh,Vh, U2Vc=II1);
  matrix C2 = interpolate (Wh,Vh, U2Vc=II2);
  matrix CC = -1*C1 + C2; // u2 - u1 > 0
  Wh cl=0; // constraints lower bounds (no upper bounds)

  // Boundary conditions
  varf vGamma([u1,u2],[v1,v2]) = on(1,2,3,4,u1=1,u2=1);
  real[int] onGamma=vGamma(0,Vh);
  Vh [ub1,ub2]=[g1,g2];
  Vh [lb1,lb2]=[g1,g2];
  ub1[] = onGamma ? ub1[] : 1e19 ; // Unbounded in interior
  lb1[] = onGamma ? lb1[] : -1e19 ;

```

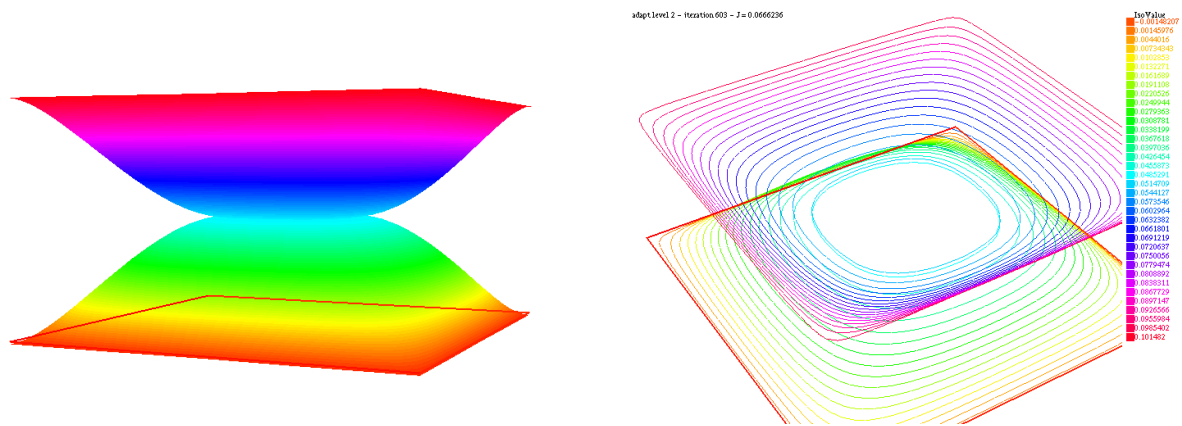


Figure 8.1: Numerical Approximation of the Variational Inequality

```

Vh [u1,u2]=[0,0]; // starting point

IPOPT([b,A],CC,u1[],lb=lb1[],ub=ub1[],clb=c1[]);

plot(u1,u2,wait=1,nbiso=60,dim=3);
if(iter > 1) break;
Th= adaptmesh(Th,[u1,u2],err=0.004,nbvx=100000);
}

```

8.5 3D constrained minimal surface with IPOPT

8.5.1 Area and volume expressions

This example aimed at numerically solving some constrained minimal surface problems with the IPOPT algorithm. We restrain to C^k ($k \geq 1$), closed, spherically parametrizable surfaces, i.e. surfaces S such that :

$$\exists \rho \in C^k([0, 2\pi] \times [0, \pi]) | S = \left\{ X = \begin{pmatrix} \rho(\theta, \phi) \\ 0 \\ 0 \end{pmatrix}, (\theta, \phi) \in [0, 2\pi] \times [0, \pi] \right\}$$

Where the components are expressed in the spherical coordinate system. Let's call Ω the $[0, 2\pi] \times [0, \pi]$ angular parameters set. In order to exclude self crossing and opened shapes, the following assumptions upon ρ are made :

$$\rho \geq 0 \text{ and } \forall \phi, \rho(0, \phi) = \rho(2\pi, \phi)$$

For a given function ρ the first fundamental form (the metric) of the defined surface has the following matrix representation :

$$G = \begin{pmatrix} \rho^2 \sin^2(\phi) + (\partial_\theta \rho)^2 & \partial_\theta \rho \partial_\phi \rho \\ \partial_\theta \rho \partial_\phi \rho & \rho^2 + (\partial_\phi \rho)^2 \end{pmatrix} \quad (8.8)$$

This metric is used to express the area of the surface. Let $g = \det(G)$, then we have :

$$\mathcal{A}(\rho) = \int_{\Omega} \|\partial_\theta X \wedge \partial_\phi X\| = \int_{\Omega} \sqrt{g} = \int_{\Omega} \sqrt{\rho^2 (\partial_\theta \rho)^2 + \rho^4 \sin^2(\phi) + \rho^2 (\partial_\phi \rho)^2 \sin^2(\phi)} d\theta d\phi \quad (8.9)$$

The volume of the space enclosed within the shape is easier to express :

$$\mathcal{V}(\rho) = \int_{\Omega} \int_0^{\rho(\theta, \phi)} r^2 \sin(\phi) dr d\theta d\phi = \frac{1}{3} \int_{\Omega} \rho^3 \sin(\phi) d\theta d\phi \quad (8.10)$$

8.5.2 Derivatives

In order to use a newton based interior point optimization algorithm, one must be able to evaluate the derivatives of \mathcal{A} and \mathcal{V} with respect to ρ . Concerning the area we have the following result :

$$\forall v \in C^1(\Omega) , \quad \langle d\mathcal{A}(\rho), v \rangle = \int_{\Omega} \frac{1}{2} \frac{d\bar{g}(\rho)(v)}{\sqrt{g}} d\theta d\phi$$

Where \bar{g} is the application mapping the $(\theta, \phi) \mapsto g(\theta, \phi)$ scalar field to ρ . This leads to the following expression, easy to transpose in a freefem script using :

$$\begin{aligned} \forall v \in C^1(\Omega) , \quad \langle d\mathcal{A}(\rho), v \rangle &= \int_{\Omega} (2\rho^3 \sin^2(\phi) + \rho(\partial_{\theta}\rho)^2 + \rho(\partial_{\phi}\rho)^2 \sin^2(\phi)) v \\ &+ \int_{\Omega} \rho^2 \partial_{\theta}\rho \partial_{\theta}v + \rho^2 \partial_{\phi}\rho \sin^2(\phi) \partial_{\phi}v \end{aligned} \quad (8.11)$$

With a similar approach, one can derive an expression for second order derivatives. Though computing no specific difficulties, the detailed calculus are tedious, the result is that these derivatives can be write using a 3×3 matrix \mathbf{B} whose coefficients are expressed in term of ρ and its derivatives with respect to θ and ϕ , such that :

$$\forall (w, v) \in C^1(\Omega) , \quad d^2\mathcal{A}(\rho)(w, v) = \int_{\Omega} \begin{pmatrix} w & \partial_{\theta}w & \partial_{\phi}w \end{pmatrix} \mathbf{B} \begin{pmatrix} v \\ \partial_{\theta}v \\ \partial_{\phi}v \end{pmatrix} d\theta d\phi \quad (8.12)$$

Deriving the volume function derivatives is again an easier task. We immediately get the following expressions :

$$\begin{aligned} \forall v , \quad \langle d\mathcal{V}(\rho), v \rangle &= \int_{\Omega} \rho^2 \sin(\phi) v d\theta d\phi \\ \forall w, v , \quad d^2\mathcal{V}(\rho)(w, v) &= \int_{\Omega} 2\rho \sin(\phi) wv d\theta d\phi \end{aligned} \quad (8.13)$$

8.5.3 The problem and its script :

The whole code is available in `IpoptMinSurfVol.edp`. We propose to solve the following problem :

Example 8.4 *Given a positive function ρ_{object} piecewise continuous, and a scalar $\mathcal{V}_{\text{max}} > \mathcal{V}(\rho_{\text{object}})$, find ρ_0 such that :*

$$\rho_0 = \underset{\rho \in C^1(\Omega)}{\operatorname{argmin}} \mathcal{A}(\rho) , \quad \text{s.t. } \rho_0 \geq \rho_{\text{object}} \text{ and } \mathcal{V}(\rho_0) \leq \mathcal{V}_{\text{max}}$$

If ρ_{object} is the spherical parametrization of the surface of a 3-dimensional object (domain) \mathcal{O} , it can be interpreted as finding the surface with minimal area enclosing the object with a given maximal volume. If \mathcal{V}_{max} is close to $\mathcal{V}(\rho_{\text{object}})$, so should be ρ_0 and ρ_{object} . With higher values of \mathcal{V}_{max} , ρ should be closer to the unconstrained minimal surface surrounding \mathcal{O} which is obtained as soon as

$\mathcal{V}_{\max} \geq \frac{4}{3}\pi\|\rho_{\text{object}}\|_{\infty}^3$ (sufficient but not necessary).

It also could be interesting to solve the same problem with the constraint $\mathcal{V}(\rho_0) \geq \mathcal{V}_{\min}$ which lead to a sphere when $\mathcal{V}_{\min} \geq \frac{1}{6}\pi\text{diam}(\mathcal{O})^3$ and move toward the solution of the unconstrained problem as \mathcal{V}_{\min} decreases.

We start by meshing the domain $[0, 2\pi] \times [0, \pi]$, then a periodic P1 finite elements space is defined.

```
load "msh3";
load "medit";
load "ff-Ipopt";

int np=40; // initial mesh quality parameter
mesh Th = square(2*np, np, [2*pi*x, pi*y]);

fespace Vh(Th, P1, periodic=[[2,y], [4,y]]);
Vh startshape=5; // initial shape
```

We create some finite element functions whose underlying arrays will be used to store the values of dual variables associated to all the constraints in order to reinitialize the algorithm with it in the case where we use mesh adaptation. Doing so, the algorithm will almost restart at the accuracy level it reached before mesh adaptation, thus saving many iterations.

```
Vh uz=1., lz=1.; // Simple bounds dual variable
real[int] lm=[1]; // dual variable for volume constraint
```

Then, follows the mesh adaptation loop, and a rendering function, Plot3D, using 3D mesh to display the shape it is passed with medit (the movemesh23 procedure often crashes when called with ragged shapes).

```
int nadapt=1;
for(int kkk=0; kkk<nadapt; ++kkk) // Mesh adaptation loop
{
    int iter=0; // iterations count
    func sin2 = square(sin(y)); // a function that will be often used

    func int Plot3D(real[int] &rho, string cmm, bool ffplot) {...} // see the .edp file
```

Here are the functions related to the area computation and its shape derivative, according to equations 8.9 and 8.11 :

```
func real Area(real[int] &X)
{
    Vh rho;
    rho[] = X;
    Vh rho2 = square(rho);
    Vh rho4 = square(rho2);
    real res = int2d(Th) ( sqrt( rho4*sin2
                                + rho2*square(dx(rho))
                                + rho2*sin2*square(dy(rho)) )
                            );
    ++iter;
```

```

    plot(rho, ... /*some parameters*/ ... );
    return res;
}

func real[int] GradArea(real[int] &X)                                // The gradient
{
    Vh rho,rho2;
    rho[] = X;
    rho2[] = square(X);
    Vh sqrtPsi,alpha;                                                // Psi is actually det(G)
    {                                                                // some optimizations
        Vh dxrho2 = dx(rho)*dx(rho), dyrho2 = dy(rho)*dy(rho);
        sqrtPsi = sqrt( rho2*rho2*sin2 + rho2*dxrho2 + rho2*dyrho2*sin2 );
        alpha = 2.*rho2*rho*sin2 + rho*dxrho2 + rho*dyrho2*sin2;
    }
    varf dSurface(u,v) =
        int2d(Th) (1./sqrtPsi*(alpha*v+rho2*dx(rho)*dx(v)+rho2*dy(rho)*sin2*dy(v)));
    real[int] grad = dSurface(0,Vh);
    return grad;
}

```

The function returning the hessian of the area for a given shape is a bit blurry, thus we won't show here all of equation 8.12 coefficients definition, they can be found in the edp file.

```

matrix hessianA;                                                    // The global matrix buffer

func matrix HessianArea(real[int] &X)
{
    Vh rho,rho2;
    rho[] = X;
    rho2 = square(rho);
    Vh sqrtPsi,sqrtPsi3,C00,C01,C02,C11,C12,C22,A;
    {
        ...                                                         // definition of the above functions
    }
    varf d2Area(w,v) =
        int2d(Th) (
            1./sqrtPsi * (A*w*v + 2*rho*dx(rho)*dx(w)*v + 2*rho*dx(rho)*w*dx(v)
                + 2*rho*dy(rho)*sin2*dy(w)*v + 2*rho*dy(rho)*sin2*w*dy(v)
                + rho2*dx(w)*dx(v) + rho2*sin2*dy(w)*dy(v))
            + 1./sqrtPsi3 * (C00*w*v + C01*dx(w)*v + C01*w*dx(v) + C02*dy(w)*v
                + C02*w*dy(v) + C11*dx(w)*dx(v)
                + C12*dx(w)*dy(v) + C12*dy(w)*dx(v) + C22*dy(w)*dy(v))
        );
    hessianA = d2Area(Vh,Vh);
    return hessianA;
}

```

And the volume related functions :

```

func real Volume(real[int] &X)
{
    Vh rho;
    rho[] = X;
    Vh rho3=rho*rho*rho;
    real res = 1./3.*int2d(Th) (rho3*sin(y));
}

```

```

    return res;
}

func real[int] GradVolume(real[int] &X)
{
    Vh rho;
    rho[]=X;
    varf dVolume(u,v) = int2d(Th) (rho*rho*sin(y)*v);
    real[int] grad = dVolume(0,Vh);
    return grad;
}

matrix hessianV; // buffer
func matrix HessianVolume(real[int] &X)
{
    Vh rho;
    rho[]=X;
    varf d2Volume(w,v) = int2d(Th) (2*rho*sin(y)*v*w);
    hessianV = d2Volume(Vh,Vh);
    return hessianV;
}

```

If we want to use the volume as a constraint function we must wrap it and its derivatives in some FreeFem++ functions returning the appropriate types. It is not done in the above functions in case where one wants to use it as fitness function. The lagrangian hessian also have to be wrapped since the Volume is not linear with respect to ρ , it has some non-null second order derivatives.

```

func real[int] ipVolume(real[int] &X) {real[int] vol = [Volume(X)]; return vol;}

matrix mdV; // buffer
func matrix ipGradVolume(real[int] &X)
{
    // transforms a vector into a sparse matrix
    real[int,int] dvol(1,Vh.ndof);
    dvol(0,:)=GradVolume(X);
    mdV=dvol;
    return mdV;
}

matrix Hlagrangian; // buffer
func matrix ipHessianLag(real[int] &X,real objfact,real[int] &lambda)
{
    Hlagrangian = objfact*HessianArea(X) + lambda[0]*HessianVolume(X);
    return Hlagrangian;
}

```

The ipGradVolume function could bring some troubles during the optimization process because the gradient vector is transformed in a sparse matrix, so any null coefficient will be discarded. We are here obliged to give IPOPT the structure by hand and use the checkindex named-parameter to avoid bad indexing during copies. This gradient is actually dense, there is no reason for some components to be constantly zero :

```

// sparse structure of a dense vector
int[int] gvi(Vh.ndof),gvj=0:Vh.ndof-1;
gvi=0; // only one line

```

These two arrays will be passed to IPOPT with `struct jacc=[gvi,gvj]`. The last remaining things are the bounds definition. The simple lower bounds must be equal to the components of the P1 projection of ρ_{object} . And we choose $\alpha \in [0, 1]$ to set \mathcal{V}_{\max} to $(1 - \alpha)\mathcal{V}(\rho_{object}) + \alpha \frac{4}{3}\pi \|\rho_{object}\|_{\infty}^3$:

```

real e=0.1,r0=0.25,rr=2-r0;
real E=1./(e*e),RR=1./(rr*rr);

                                                    // An indented disc
func disc1 = sqrt(1./(RR+(E-RR)*cos(y)*cos(y)))*(1+0.1*cos(9*x));
                                                    // Almost a standard disc
func disc2 = sqrt(1./(RR+(E-RR)*cos(x)*cos(x)*sin(2)));
Vh lb = max(disc1, disc2);
real Vobj = Volume(lb[]);
real Vnvc = 4./3.*pi*pow(lb[].linfty,3);
real alpha=0.1;
Plot3D(lb[],"object_inside",0);
real[int] clb=0.,cub=[(1-alpha)*Vobj + alpha*Vnvc];

```

Calling IPOPT :

```

IPOPT(Area,GradArea,ipHessianLag,
      ipVolume,ipGradVolume,rc[],
      ub=ub[],lb=lb[],clb=clb,cub=cub,
      checkindex=1,struct jacc=[gvi,gvj],
      maxiter=kkk<nadapt-1 ? 40:150,
      warmstart=kkk,lm=lm,uz=uz[],lz=lz[],
      tol=0.00001);
Plot3D(rc[],"Shape_at_"+kkk,0);

```

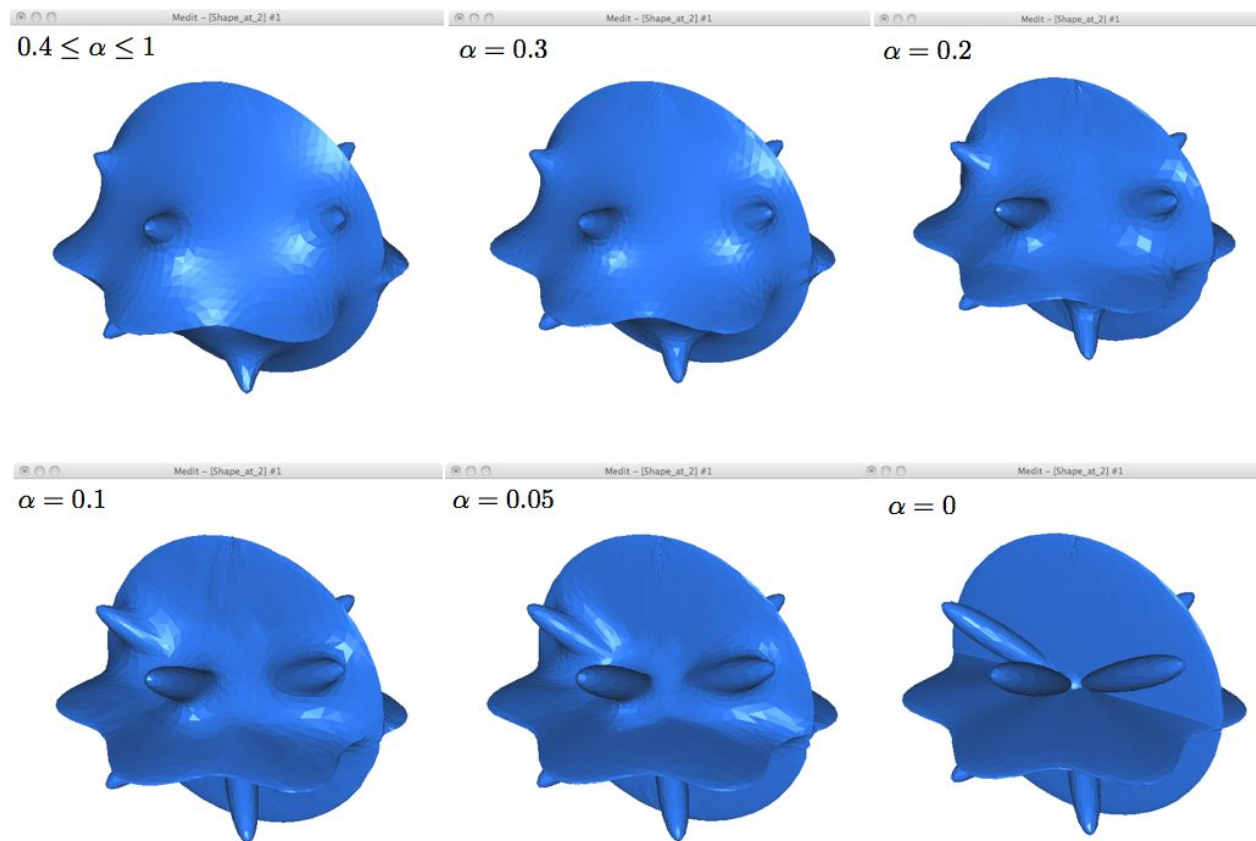
At last, before closing the mesh adaptation loop, we have to perform the said adaptation. The mesh is adaptated with respect to the $X = (\rho, 0, 0)$ (in spherical coordinates) vector field, not directly with respect to ρ , otherwise the true curvature of the 3D-shape would not be well taken into account.

```

if (kkk<nadapt-1)
{
  Th = adaptmesh(Th,
    rc*cos(x)*sin(y),
    rc*sin(x)*sin(y),
    rc*cos(y),
    nbvx=50000,
    periodic=[[2,y],[4,y]]);
  plot(Th);
  startshape = rc;
  uz=uz;
  lz=lz;
}
}

```

Here are some pictures of the resulting surfaces obtained for decreasing values of α (and a slightly more complicated object than two orthogonal discs). We get back the enclosed object when $\alpha = 0$:



8.6 The nlOpt optimizers

The `ff-NlOpt` package provides a FreeFem interface to the free/open-source library for nonlinear optimization, thus easing the use of several different free optimization (constrained or not) routines available online along with the PDE solver. All the algorithms are well documented in [42], thus no exhaustive informations concerning their mathematical specificities will be found here and we will focus on the way they are called in a FreeFem script. One needing detailed informations about these algorithms should visit the said cite where a description of each of them is given, as well as many bibliographical links. Most of the gradient based algorithm of `nlOpt` uses a full matrix approximation of the hessian, so if you're planing to solve a large scale problem, our advise would be to use the IPOPT optimizer which definitely surpass them. Finally, an examples of use can be found in the `examples++-load/` directory under the name `VarIneq2.edp`

All the `nlOpt` features are called that way :

```
load "ff-NlOpt"
...           // define J, u, and maybe grad(J), some constraints etc...
real min = nloptXXXXX(J,u,                          // unavoidable part
                    grad = <name of grad(J)> ,        // if needed
                    lb =                               // lower bounds array
                    ub=                               // upper bounds array
                    ...                               // some optional arguments :
                    // constraints functions names,
```



```

//      stopping criterions,
//      algo.  specific parameters,
//      etc...

);

```

XXXXXX refers to the algorithm tag (not necessarily 6 characters long). `u` is the starting position (a `real[int]` type array) which will be overwritten by the algorithm, the value at the end being the found argmin. And as usual, `J` is a function taking a `real[int]` type array as argument and returning a `real`. `grad`, `lb` and `ub` are "half-optionnal" arguments, in the sense that they are obligatory for some routines but not all.

The possible optional named parameters are the following, note that they are not used by all algorithms (some does not supports constraints, or a type of constraints, some are gradient-based and other are derivative free, etc...). One can refer to the table after the parameters description to check which are the named parameters supported by a specific algorithm. Using an unsupported parameter will not stop the compiler work and seldom breaks runtime, it will just be ignored. That said, when it is obvious you are missusing a routine, you will get a warning message at runtime (for exemple if you pass a gradient to a derivative free algorithm, or set the population of a non-genetic one, etc...). In the following description, n stands for the dimension of the search space.

Half-optional parameters :

- grad=** The name of the function which computes the gradient of the cost function (prototype should be `real[int] → real[int]`, both argument and result should have the size n). This is needed as soon as a gradient-based method is involved, ignored if defined in a derivative free context.
- lb/ub =** Lower and upper bounds arrays (`real[int]` type) of size n . Used to define the bounds within which the search variable is allowed to move. Needed for some algorithms, optional or unsupported for others.
- subOpt :** Only enabled for the Augmented Lagrangian and MLSL method who need a sub-optimizer in order to work. Just pass the tag of the desired local algorithm with a string.

Constraints related parameters (optional - unused if not specified):

- IConst/EConst :** Allows to pass the name of a function implementing some inequality (resp. equality) constraints on the search space. The function type must be `real[int] → real[int]` where the size of the returned array is equal to the number of constraints (of the same type - it means that all the constraints are computed in one vectorial function). In order to mix inequality and equality constraints in a same minimization attempt, two vectorial functions have to be defined and passed. See example ?? for more details about how these constraints have to be implemented.
- gradIConst/gradEConst :** Use to provide the inequality (resp. equality) constraints gradient. These are `real[int] → real[int,int]` type functions. Assuming we have defined a constraint function (either inequality or equality) with p constraints, the size of the matrix returned by its associated gradient must be $p \times n$ (the i -th line of the matrix is the gradient of the i -th constraint). It is needed in a gradient-based context as soon as an inequality or equality constraint function is passed to the optimizer and ignored in all other cases.

tolIConst/tolEConst : Tolerance values for each constraint. This is an array of size equal to the number of inequality (resp. equality) constraints. Default value is set to 10^{-12} for each constraint of any type.

Stopping criteria :

stopFuncValue : Makes the algorithm end when the objective function reaches this real value.

stopRelXTol : Stops the algorithm when the relative moves in each direction of the search space is smaller than this real value.

stopAbsXTol : Stops the algorithm when the moves in each direction of the search space is smaller than the corresponding value in this `real[int]` array.

stopRelFTol : Stops the algorithm when the relative variation of the objective function is smaller than this real value.

stopAbsFTol : Stops the algorithm when the variation of the objective function is smaller than this real value.

stopMaxFEval : Stops the algorithm when the number of fitness evaluations reaches this integer value.

stopTime : Stops the algorithm when the optimization time in second exceeds this real value. This is not a strict maximum: the time may exceed it slightly, depending upon the algorithm and on how slow your function evaluation is.

Note that when an AUGLAG or MLSL method is used, the meta-algorithm and the sub-algorithm may have different termination criteria. Thus, for algorithms of this kind, the following named parameters has been defined (just adding the SO prefix - for Sub-Optimizer) to set the ending condition of the sub-algorithm (the meta one uses the ones above) : `SOSTopFuncValue`, `SOSTopRelXTol`, and so on... If these ones are not used, the sub-optimizer will use those of the master routine.

Other named parameters :

popSize : integer used to change the size of the sample for stochastic search methods. Default value is a peculiar heuristic to the chosen algorithm.

SOPopSize : Same as above, but when the stochastic search is passed to a meta-algorithm.

nGradStored : The number (integer type) of gradients to "remember" from previous optimization steps: increasing this increases the memory requirements but may speed convergence. It is set to a heuristic value by default. If used with AUGLAG or MLSL, it will only affect the given subsidiary algorithm.

The following table sums up the main characteristics of each algorithm, providing the more important information about which features are supported by which algorithm and what are the unavoidable arguments they need. More details can be found in [42].

Id Tag	Full Name	Bounds	Gradient -Based	Stochastic	Constraints		Sub- Opt
					Equality	Inequality	
DIRECT	Dividing rectangles	●					
DIRECTL	Locally biased dividing rectangles	●					
DIRECTL Rand	Randomized locally biased dividing rectangles	●					
DIRECTNoScal	Dividing rectangles - no scaling	●					
DIRECTLNoScal	Locally biased dividing rectangles - no scaling	●					
DIRECTL RandNoScal	Randomized locally biased dividing rectangles - no scaling	●					
OrigDIRECT	Original Glabonsky's dividing rectangles	●				✓	
OrigDIRECTL	Original Glabonsky's locally biased dividing rectangles	●				✓	
StoGO	Stochastic(?) Global Optimization	●	●				
StoGO Rand	Randomized Stochastic(?) Global Optimization	●	●				
LBFGS	Low-storage BFGS		●				
PRAXIS	Principal AXIS	✓					
Var1	Rank-1 shifted limited-memory variable-metric		●				
Var2	Rank-2 shifted limited-memory variable-metric		●				
TNewton	Truncated Newton		●				
TNewtonRestart	Steepest descent restarting truncated Newton		●				
TNewtonPrecond	BFGS preconditionned truncated Newton		●				
TNewtonRestartPrecond	BFGS preconditionned truncated Newton with steepest descent restarting		●				
CRS2	Controlled random search with local mutation	✓		●			
MMA	Method of moving asymptots	✓	●			✓	
COBYLA	Constrained optimization by linear approximations	✓			✓	✓	
NEWUOA	NEWUOA						
NEWUOABound	NEWUOA for bounded optimization	✓					
NelderMead	Nelder-Mead simplex	✓					
Sbplx	Subplex	✓					
BOBYQA	BOBYQA	✓					
ISRES	Improved stochastic ranking evolution strategy	✓		●	✓	✓	
SLSQP	Sequential least-square quadratic programming	✓	●		✓	✓	
MLSL	Multi-level single-linkage	✓	●	●			●
MLSL LDS	Low discrepancy multi-level single-linkage	✓	●	●			●
AUGLAG	Constraints augmented lagrangian	✓	●		✓	✓	●
AUGLAGEQ	Equality constraints augmented lagrangian	✓	●		✓	✓	●

Legend :

- ✓ Supported and optional
- ✓ Should be supported and optional, may lead to weird behaviour though.
- ✓ Intrinsic characteristic of the algorithm which then need one or more unavoidable parameter to work (for stochastic algorithm, the population size always have a default value, they will then work if it is omitted)
- For routines with subsidiary algorithms only, indicates that the corresponding feature will depend on the chosen sub-optimizer.

8.7 Optimization with MPI

The only quick way to use the previously presented algorithms on a parallel architecture lies in parallelizing the used cost function (which is in most real life case, the expensive part of the algorithm). Somehow, we provide a parallel version of the CMA-ES algorithm. The parallelization principle is the trivial one of evolving/genetic algorithms : at each iteration the cost function has to be evaluated N times without any dependence at all, these N calculus are then equally distributed to each processes. Calling the MPI version of CMA-ES is nearly the same as calling its sequential version (a complete example of use can be found in the `cmaes-mpi-VarIneq.edp` file):

```
load "mpi-cmaes"
...
// define J, u and all here
real min = cmaesMPI(J,u,stopTolFun=1e-6,stopMaxIter=3000);
cout << "minimal value is " << min << " for u = " << u << endl;
```

If the population size is not changed using the `popsize` parameter, it will use the heuristic value slightly changed to be equal to the closest greater multiple of the size of the communicator used by the optimizer. The FreeFem `mpicommworld` is used by default. The user can specify his own MPI communicator with the named parameter " `comm=`", see the MPI section of this manual for more informations about communicators in FreeFem++ .

Chapter 9

Mathematical Models

Summary *This chapter goes deeper into a number of problems that FreeFem++ can solve. It is a complement to chapter 3 which was only an introduction. Users are invited to contribute to make this data base of problems grow.*

9.1 Static Problems

9.1.1 Soap Film

Our starting point here will be the mathematical model to find the shape of **soap film** which is glued to the ring on the xy -plane

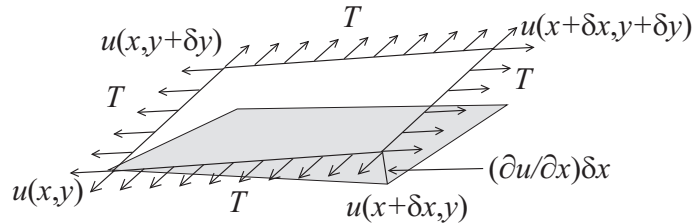
$$C = \{(x, y); x = \cos t, y = \sin t, 0 \leq t \leq 2\pi\}.$$

We assume the shape of the film is described by the graph $(x, y, u(x, y))$ of the vertical displacement $u(x, y)$ ($x^2 + y^2 < 1$) under a vertical pressure p in terms of force per unit area and an initial tension μ in terms of force per unit length.

Consider the “small plane” ABCD, A: $(x, y, u(x, y))$, B: $(x, y, u(x + \delta x, y))$, C: $(x, y, u(x + \delta x, y + \delta y))$ and D: $(x, y, u(x, y + \delta y))$. Denote by $\mathbf{n}(x, y) = (n_x(x, y), n_y(x, y), n_z(x, y))$ the normal vector of the surface $z = u(x, y)$. We see that the vertical force due to the tension μ acting along the edge AD is $-\mu n_x(x, y)\delta y$ and the the vertical force acting along the edge AD is

$$\mu n_x(x + \delta x, y)\delta y \simeq \mu \left(n_x(x, y) + \frac{\partial n_x}{\partial x} \delta x \right) (x, y)\delta y.$$

Similarly, for the edges AB and DC we have

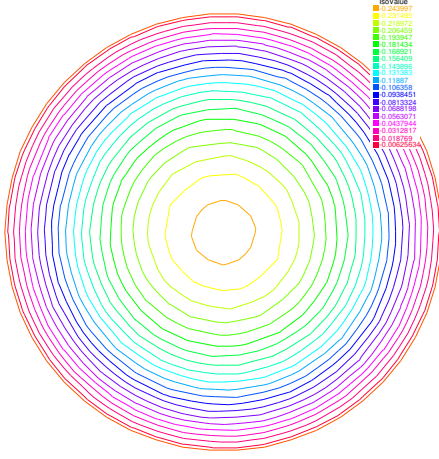
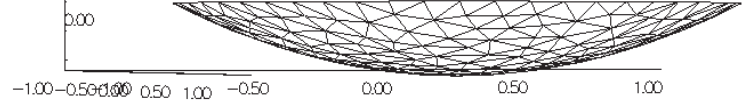


$$-\mu n_y(x, y)\delta x, \quad \mu (n_y(x, y) + \partial n_y / \partial y) (x, y)\delta x.$$


```

30 : plot(err,value=true,wait=true);
31 : cout << "error L2=" << sqrt(int2d(disk)(err^2)) << endl;
32 : cout << "error H10=" << sqrt(int2d(disk)((dx(u)-x/2)^2)
33 :                               + int2d(disk)((dy(u)-y/2)^2)) << endl;

```

Figure 9.1: isovalue of u Figure 9.2: a side view of u

In 19th line, the L^2 -error estimation between the exact solution u_e ,

$$\|u_h - u_e\|_{0,\Omega} = \left(\int_{\Omega} |u_h - u_e|^2 dx dy \right)^{1/2}$$

and from 20th line to 21th line, the H^1 -error seminorm estimation

$$|u_h - u_e|_{1,\Omega} = \left(\int_{\Omega} |\nabla u_h - \nabla u_e|^2 dx dy \right)^{1/2}$$

are done on the initial mesh. The results are $\|u_h - u_e\|_{0,\Omega} = 0.000384045$, $|u_h - u_e|_{1,\Omega} = 0.0375506$. After the adaptation, we have $\|u_h - u_e\|_{0,\Omega} = 0.000109043$, $|u_h - u_e|_{1,\Omega} = 0.0188411$. So the numerical solution is improved by adaptation of mesh.

9.1.2 Electrostatics

We assume that there is no current and a time independent charge distribution. Then the electric field \mathbf{E} satisfies

$$\operatorname{div} \mathbf{E} = \rho/\epsilon, \quad \operatorname{curl} \mathbf{E} = 0 \quad (9.3)$$

where ρ is the charge density and ϵ is called the permittivity of free space. From the second equation in (9.3), we can introduce the electrostatic potential such that $\mathbf{E} = -\nabla \phi$. Then we have Poisson's equation $-\Delta \phi = f$, $f = -\rho/\epsilon$. We now obtain the equipotential line which is the level curve of ϕ , when there are no charges except conductors $\{C_i\}_{1,\dots,K}$. Let us assume K conductors C_1, \dots, C_K within an enclosure C_0 . Each one is held at an electrostatic potential φ_i . We assume that the enclosure C_0 is held at potential 0. In order to know $\varphi(x)$ at any point x of the domain Ω , we must solve

$$-\Delta \varphi = 0 \quad \text{in } \Omega, \quad (9.4)$$

where Ω is the interior of C_0 minus the conductors C_i , and Γ is the boundary of Ω , that is $\sum_{i=0}^N C_i$. Here g is any function of x equal to φ_i on C_i and to 0 on C_0 . The boundary equation is a reduced form for:

$$\varphi = \varphi_i \text{ on } C_i, i = 1 \dots N, \varphi = 0 \text{ on } C_0. \quad (9.5)$$

Example 9.2 First we give the geometrical informations; $C_0 = \{(x, y); x^2 + y^2 = 5^2\}$, $C_1 = \{(x, y) : \frac{1}{0.3^2}(x - 2)^2 + \frac{1}{3^2}y^2 = 1\}$, $C_2 = \{(x, y) : \frac{1}{0.3^2}(x + 2)^2 + \frac{1}{3^2}y^2 = 1\}$. Let Ω be the disk enclosed by C_0 with the elliptical holes enclosed by C_1 and C_2 . Note that C_0 is described counterclockwise, whereas the elliptical holes are described clockwise, because the boundary must be oriented so that the computational domain is to its left.

```

//      a circle with center at (0 ,0) and radius 5
border C0(t=0,2*pi) { x = 5 * cos(t); y = 5 * sin(t); }
border C1(t=0,2*pi) { x = 2+0.3 * cos(t); y = 3*sin(t); }
border C2(t=0,2*pi) { x = -2+0.3 * cos(t); y = 3*sin(t); }

mesh Th = buildmesh(C0(60)+C1(-50)+C2(-50));
plot(Th,ps="electroMesh");
fespace Vh(Th,P1);
Vh uh,vh;
problem Electro(uh,vh) =
    int2d(Th) ( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
    + on(C0,u=0)
    + on(C1,u=1)
    + on(C2,u=-1) ;

Electro;
plot(uh,ps="electro.eps",wait=true);

```

// figure 9.3
// P1 FE-space
// unknown and test function.
// definition of the problem
// bilinear
// boundary condition on C_0
// +1 volt on C_1
// -1 volt on C_2
// solve the problem, see figure 9.4 for the solution
// figure 9.4

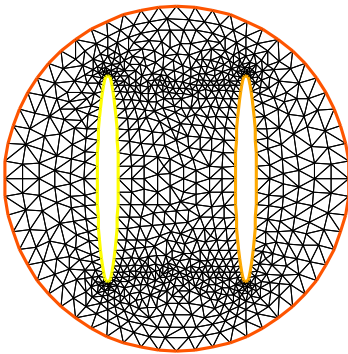


Figure 9.3: Disk with two elliptical holes

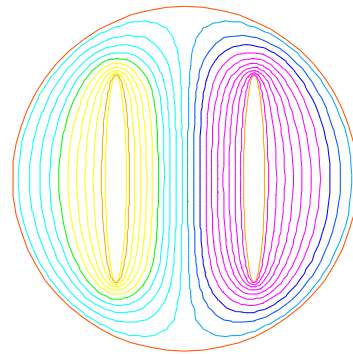


Figure 9.4: Equipotential lines, where C_1 is located in right hand side

9.1.3 Aerodynamics

Let us consider a wing profile S in a uniform flow. Infinity will be represented by a large circle Γ_∞ . As previously, we must solve

$$\Delta\varphi = 0 \quad \text{in } \Omega, \quad \varphi|_S = c, \quad \varphi|_{\Gamma_\infty} = u_{\infty 1}x - u_{\infty 2}y \quad (9.6)$$

where Ω is the area occupied by the fluid, u_∞ is the air speed at infinity, c is a constant to be determined so that $\partial_n\varphi$ is continuous at the trailing edge P of S (so-called Kutta-Joukowski condition). Lift is proportional to c . To find c we use a superposition method. As all equations in (9.6) are linear, the solution φ_c is a linear function of c

$$\varphi_c = \varphi_0 + c\varphi_1, \quad (9.7)$$

where φ_0 is a solution of (9.6) with $c = 0$ and φ_1 is a solution with $c = 1$ and zero speed at infinity. With these two fields computed, we shall determine c by requiring the continuity of $\partial\varphi/\partial n$ at the trailing edge. An equation for the upper surface of a NACA0012 (this is a classical wing profile in aerodynamics; the rear of the wing is called the trailing edge) is:

$$y = 0.17735\sqrt{x} - 0.075597x - 0.212836x^2 + 0.17363x^3 - 0.06254x^4. \quad (9.8)$$

Taking an incidence angle α such that $\tan \alpha = 0.1$, we must solve

$$-\Delta\varphi = 0 \quad \text{in } \Omega, \quad \varphi|_{\Gamma_1} = y - 0.1x, \quad \varphi|_{\Gamma_2} = c, \quad (9.9)$$

where Γ_2 is the wing profile and Γ_1 is an approximation of infinity. One finds c by solving:

$$-\Delta\varphi_0 = 0 \quad \text{in } \Omega, \quad \varphi_0|_{\Gamma_1} = y - 0.1x, \quad \varphi_0|_{\Gamma_2} = 0, \quad (9.10)$$

$$-\Delta\varphi_1 = 0 \quad \text{in } \Omega, \quad \varphi_1|_{\Gamma_1} = 0, \quad \varphi_1|_{\Gamma_2} = 1. \quad (9.11)$$

The solution $\varphi = \varphi_0 + c\varphi_1$ allows us to find c by writing that $\partial_n\varphi$ has no jump at the trailing edge $P = (1, 0)$. We have $\partial_n\varphi - (\varphi(P^+) - \varphi(P))/\delta$ where P^+ is the point just above P in the direction normal to the profile at a distance δ . Thus the jump of $\partial_n\varphi$ is $(\varphi_0|_{P^+} + c(\varphi_1|_{P^+} - 1)) + (\varphi_0|_{P^-} + c(\varphi_1|_{P^-} - 1))$ divided by δ because the normal changes sign between the lower and upper surfaces. Thus

$$c = -\frac{\varphi_0|_{P^+} + \varphi_0|_{P^-}}{(\varphi_1|_{P^+} + \varphi_1|_{P^-} - 2)}, \quad (9.12)$$

which can be programmed as:

$$c = -\frac{\varphi_0(0.99, 0.01) + \varphi_0(0.99, -0.01)}{(\varphi_1(0.99, 0.01) + \varphi_1(0.99, -0.01) - 2)}. \quad (9.13)$$

Example 9.3 *// Computation of the potential flow around a NACA0012 airfoil.
 // The method of decomposition is used to apply the Joukowski condition
 // The solution is seeked in the form psi0 + beta psi1 and beta is
 // adjusted so that the pressure is continuous at the trailing edge*

```
border a(t=0,2*pi) { x=5*cos(t); y=5*sin(t); };    //    approximates infinity

border upper(t=0,1) { x = t;
          y = 0.17735*sqrt(t)-0.075597*t
          - 0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4); }
border lower(t=1,0) { x = t;
```

```

y= -(0.17735*sqrt(t)-0.075597*t
-0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4)); }
border c(t=0,2*pi) { x=0.8*cos(t)+0.5; y=0.8*sin(t); }

wait = true;
mesh Zoom = buildmesh(c(30)+upper(35)+lower(35));
mesh Th = buildmesh(a(30)+upper(35)+lower(35));
fespace Vh(Th,P2); // P1 FE space
Vh psi0,psil,vh; // unknown and test function.
fespace ZVh(Zoom,P2);

solve Joukowski0(psi0,vh) = // definition of the problem
  int2d(Th) ( dx(psi0)*dx(vh) + dy(psi0)*dy(vh) ) // bilinear form
+ on(a,psi0=y-0.1*x) // boundary condition form
+ on(upper,lower,psi0=0);
plot(psi0);

solve Joukowski1(psil,vh) = // definition of the problem
  int2d(Th) ( dx(psil)*dx(vh) + dy(psil)*dy(vh) ) // bilinear form
+ on(a,psil=0) // boundary condition form
+ on(upper,lower,psil=1);

plot(psil);

// continuity of pressure at trailing edge
real beta = psi0(0.99,0.01)+psi0(0.99,-0.01);
beta = -beta / (psil(0.99,0.01)+ psil(0.99,-0.01)-2);

Vh psi = beta*psil+psi0;
plot(psi);
ZVh Zpsi=psi;
plot(Zpsi,bw=true);
ZVh cp = -dx(psi)^2 - dy(psi)^2;
plot(cp);
ZVh Zcp=cp;
plot(Zcp,nbiso=40);

```

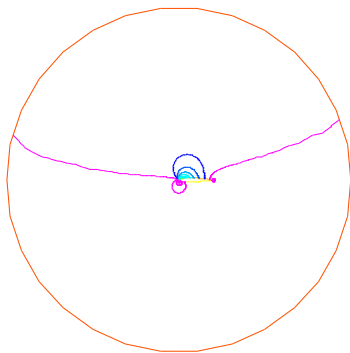


Figure 9.5: isovalue of $cp = -(\partial_x \psi)^2 - (\partial_y \psi)^2$

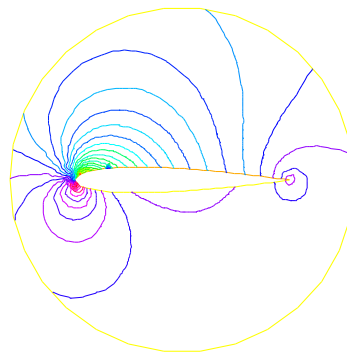


Figure 9.6: Zooming of cp

9.1.4 Error estimation

There are famous estimation between the numerical result u_h and the exact solution u of the problem 2.1 and 2.2: If triangulations $\{\mathcal{T}_h\}_{h \downarrow 0}$ is regular (see Section 5.4), then we have the estimates

$$|\nabla u - \nabla u_h|_{0,\Omega} \leq C_1 h \quad (9.14)$$

$$\|u - u_h\|_{0,\Omega} \leq C_2 h^2 \quad (9.15)$$

with constants C_1, C_2 independent of h , if u is in $H^2(\Omega)$. It is known that $u \in H^2(\Omega)$ if Ω is convex. In this section we check (9.14) and (9.15). We will pick up numerical error if we use the numerical derivative, so we will use the following for (9.14).

$$\begin{aligned} \int_{\Omega} |\nabla u - \nabla u_h|^2 dx dy &= \int_{\Omega} \nabla u \cdot \nabla (u - 2u_h) dx dy + \int_{\Omega} \nabla u_h \cdot \nabla u_h dx dy \\ &= \int_{\Omega} f(u - 2u_h) dx dy + \int_{\Omega} f u_h dx dy \end{aligned}$$

The constants C_1, C_2 are depend on \mathcal{T}_h and f , so we will find them by FreeFem++ . In general, we cannot get the solution u as a elementary functions (see Section 4.8) even if spetical functions are added. Instead of the exact solution, here we use the approximate solution u_0 in $V_h(\mathcal{T}_h, P_2)$, $h \sim 0$.

Example 9.4

```

1 : mesh Th0 = square(100,100);
2 : fespace V0h(Th0,P2);
3 : V0h u0,v0;
4 : func f = x*y; // sin(pi*x)*cos(pi*y);
5 :
6 : solve Poisson0(u0,v0) =
7 :   int2d(Th0)( dx(u0)*dx(v0) + dy(u0)*dy(v0) ) // bilinear form
8 :   - int2d(Th0)( f*v0 ) // linear form
9 :   + on(1,2,3,4,u0=0) ; // boundary condition
10 :
11 : plot(u0);
12 :
13 : real[int] errL2(10), errH1(10);
14 :
15 : for (int i=1; i<=10; i++) {
16 :   mesh Th = square(5+i*3,5+i*3);
17 :   fespace Vh(Th,P1);
18 :   fespace Ph(Th,P0);
19 :   Ph h = hTriangle; // get the size of all triangles
20 :   Vh u,v;
21 :   solve Poisson(u,v) =
22 :     int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v) ) // bilinear form
23 :     - int2d(Th)( f*v ) // linear form
24 :     + on(1,2,3,4,u=0) ; // boundary condition
25 :   V0h uu = u;
26 :   errL2[i-1] = sqrt( int2d(Th0)((uu - u0)^2) )/h[].max^2;
27 :   errH1[i-1] = sqrt( int2d(Th0)( f*(u0-2*uu+uu) ) )/h[].max;
28 : }
29 : cout << "C1 = " << errL2.max << "(" << errL2.min << ")" << endl;
30 : cout << "C2 = " << errH1.max << "(" << errH1.min << ")" << endl;

```

We can guess that $C_1 = 0.0179253(0.0173266)$ and $C_2 = 0.0729566(0.0707543)$, where the numbers inside the parentheses are minimum in calculation.


```

border a(t=0,1){x=-t+1; y=t;label=1;};
border b(t=0,1){ x=-t; y=1-t;label=2;};
border c(t=0,1){ x=t-1; y=-t;label=3;};
border d(t=0,1){ x=t; y=-1+t;label=4;};
border e(t=0,2*pi){ x=r*cos(t); y=-r*sin(t);label=0;};
int n = 10;
mesh Th= buildmesh(a(n)+b(n)+c(n)+d(n)+e(n));
plot(Th,wait=1);
real r2=1.732;
func abs=sqrt(x^2+y^2);
// warning for periodic condition:
// side a and c
// on side a (label 1)  $x \in [0,1]$  or  $x-y \in [-1,1]$ 
// on side c (label 3)  $x \in [-1,0]$  or  $x-y \in [-1,1]$ 
// so the common abscissa can be respectively  $x$  and  $x+1$ 
// or you can try curviline abscissa  $x-y$  and  $x-y$ 
// 1 first way
// fespace Vh(Th,P2,periodic=[[2,1+x],[4,x],[1,x],[3,1+x]]);
// 2 second way
fespace Vh(Th,P2,periodic=[[2,x+y],[4,x+y],[1,x-y],[3,x-y]]);

Vh uh,vh;

func f=(y+x+1)*(y+x-1)*(y-x+1)*(y-x-1);
real intf = int2d(Th)(f);
real mTh = int2d(Th)(1);
real k = intf/mTh;
problem laplace(uh,vh) =
  int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) ) + int2d(Th)( (k-f)*vh ) ;
laplace;
plot(uh,wait=1,ps="perio4.eps");

```

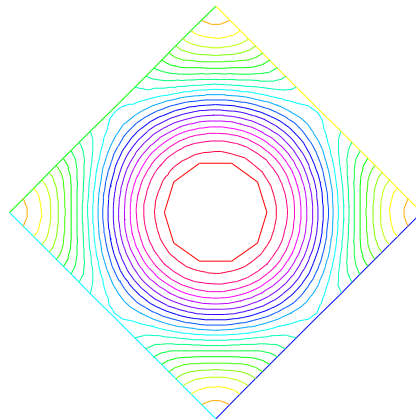


Figure 9.8: The isovalue of solution u for $\Delta u = ((y+x)^2 + 1)((y-x)^2 + 1) - k$, in Ω and $\partial_n u = 0$ on hole, and with two periodic boundary condition on external border

An other example with no equal border, just to see if the code works.

Example 9.7 (periodic4bis.edp)

```

//      irregular boundary condition.
//      to build border AB
macro LINEBORDER(A,B,lab) border A#B(t=0,1){real t1=1.-t;
  x=A#x*t1+B#x*t; y=A#y*t1+B#y*t; label=lab;} //      EOM
//      compute ||AB|| a=(ax,ay) et B=(bx,by)
macro dist(ax,ay,bx,by) sqrt(square((ax)-(bx))+ square((ay)-(by))) //      EOM
macro Grad(u) [dx(u),dy(u)] //      EOM

real Ax=0.9,Ay=1;          real Bx=2,By=1;
real Cx=2.5,Cy=2.5;        real Dx=1,Dy=2;
real gx = (Ax+Bx+Cx+Dx)/4.; real gy = (Ay+By+Cy+Dy)/4.;

LINEBORDER(A,B,1)
LINEBORDER(B,C,2)
LINEBORDER(C,D,3)
LINEBORDER(D,A,4)

int n=10;

real l1=dist(Ax,Ay,Bx,By);
real l2=dist(Bx,By,Cx,Cy);
real l3=dist(Cx,Cy,Dx,Dy);
real l4=dist(Dx,Dy,Ax,Ay);
func s1=dist(Ax,Ay,x,y)/l1; //      abscisse on AB = ||AX||/||AB||
func s2=dist(Bx,By,x,y)/l2; //      abscisse on BC = ||BX||/||BC||
func s3=dist(Cx,Cy,x,y)/l3; //      abscisse on CD = ||CX||/||CD||
func s4=dist(Dx,Dy,x,y)/l4; //      abscisse on DA = ||DX||/||DA||

mesh Th=buildmesh(AB(n)+BC(n)+CD(n)+DA(n),fixeborder=1); //

verbosity=6; //      to see the abscisse value pour the periodic condition.
fespace Vh(Th,P1,periodic=[[1,s1],[3,s3],[2,s2],[4,s4]]);
verbosity=1;
Vh u,v;
real cc=0;
cc= int2d(Th)((x-gx)*(y-gy)-cc)/Th.area;
cout << " compatibility =" << int2d(Th)((x-gx)*(y-gy)-cc) <<endl;

solve Poission(u,v)=int2d(Th)(Grad(u)'*Grad(v)+ 1e-10*u*v)
  -int2d(Th)(10*v*((x-gx)*(y-gy)-cc));
plot(u,wait=1,value=1);

```

Example 9.8 (Period-Poisson-cube-ballon.edp)

```

verbosity=1;
load "msh3"
load "tetgen"
load "medit"

bool buildTh=0;

```

```

mesh3 Th;
try {           //    a way to build one time the mesh and read if the file exist.
    Th=readmesh3("Th-hex-sph.mesh");
}
catch(...) { buildTh=1;}
if( buildTh ){
    ...
    put the code example page //    5.11.1143
    without the first line
}

fespace Ph(Th,P0);
verbosity=50;
fespace Vh(Th,P1,periodic=[[3,x,z],[4,x,z],[1,y,z],[2,y,z],[5,x,y],[6,x,y]]); //
back and front
verbosity=1;
Ph reg=region;

cout << "  centre = " << reg(0,0,0) << endl;
cout << "  exterieur = " << reg(0,0,0.7) << endl;

macro Grad(u) [dx(u),dy(u),dz(u)] //    EOM

Vh uh,vh;
real x0=0.3,y0=0.4,z0=0.6;
func f= sin(x*2*pi+x0)*sin(y*2*pi+y0)*sin(z*2*pi+z0);
real gn = 1.;
real cf= 1;
problem P(uh,vh)=
    int3d(Th,1) ( Grad(uh)'*Grad(vh)*100)
    + int3d(Th,2) ( Grad(uh)'*Grad(vh)*2)
    + int3d(Th) (vh*f)
    ;
    P;
plot(uh,wait=1, nbiso=6);
medit("  uh ",Th, uh);

```

9.1.6 Poisson Problems with mixed boundary condition

Here we consider the Poisson equation with mixed boundary conditions: For given functions f and g , find u such that

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega \\ u &= g & \text{on } \Gamma_D, \quad \partial u / \partial n = 0 & \text{on } \Gamma_N \end{aligned} \quad (9.16)$$

where Γ_D is a part of the boundary Γ and $\Gamma_N = \Gamma \setminus \overline{\Gamma_D}$. The solution u has the singularity at the points $\{\gamma_1, \gamma_2\} = \overline{\Gamma_D} \cap \overline{\Gamma_N}$. When $\Omega = \{(x, y); -1 < x < 1, 0 < y < 1\}$, $\Gamma_N = \{(x, y); -1 \leq x < 0, y = 0\}$, $\Gamma_D = \partial\Omega \setminus \Gamma_N$, the singularity will appear at $\gamma_1 = (0, 0)$, $\gamma_2 = (-1, 0)$, and u has the expression

$$u = K_i u_S + u_R, \quad u_R \in H^2(\text{near } \gamma_i), \quad i = 1, 2$$

with a constants K_i . Here $u_S = r_j^{1/2} \sin(\theta_j/2)$ by the local polar coordinate (r_j, θ_j) at γ_j such that $(r_1, \theta_1) = (r, \theta)$. Instead of polar coordinate system (r, θ) , we use that $r = \sqrt{x^2 + y^2}$ and $\theta = \text{atan2}(y, x)$ in FreeFem++ .

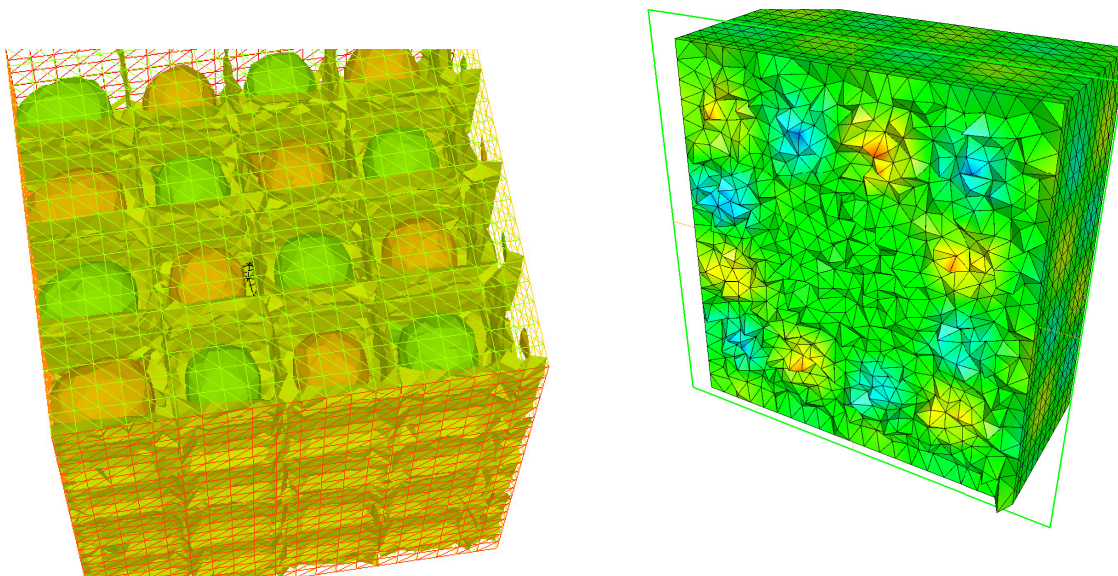


Figure 9.10:

Figure 9.9: view of the surface isovalue of periodic solution uh

view a the cut of the solution uh with fmedit

Example 9.9 Assume that $f = -2 \times 30(x^2 + y^2)$ and $g = u_e = 10(x^2 + y^2)^{1/4} \sin([\tan^{-1}(y/x)]/2) + 30(x^2 y^2)$, where u_e is the exact solution.

```

1 : border N(t=0,1) { x=-1+t; y=0; label=1; };
2 : border D1(t=0,1){ x=t; y=0; label=2;};
3 : border D2(t=0,1){ x=1; y=t; label=2; };
4 : border D3(t=0,2){ x=1-t; y=1; label=2;};
5 : border D4(t=0,1) { x=-1; y=1-t; label=2; };
6 :
7 : mesh T0h = buildmesh(N(10)+D1(10)+D2(10)+D3(20)+D4(10));
8 : plot(T0h,wait=true);
9 : fespace V0h(T0h,P1);
10 : V0h u0, v0;
11 :
12 : func f=-2*30*(x^2+y^2); // given function
13 : // the singular term of the solution is K*us (K: constant)
14 : func us = sin(atan2(y,x)/2)*sqrt( sqrt(x^2+y^2) );
15 : real K=10.;
16 : func ue = K*us + 30*(x^2*y^2);
17 :
18 : solve Poisson0(u0,v0) =
19 :     int2d(T0h)( dx(u0)*dx(v0) + dy(u0)*dy(v0) ) // bilinear form
20 :     - int2d(T0h)( f*v0 ) // linear form
21 :     + on(2,u0=ue) ; // boundary condition
22 :
23 : // adaptation by the singular term
24 : mesh Th = adaptmesh(T0h,us);
25 : for (int i=0;i< 5;i++)
26 : {
27 :     mesh Th=adaptmesh(Th,us);
28 : } ;
29 :

```



```

30 : fespace Vh(Th, P1);
31 : Vh u, v;
32 : solve Poisson(u,v) =
33 :     int2d(Th) ( dx(u)*dx(v) + dy(u)*dy(v) )           //    bilinear form
34 :     - int2d(Th) ( f*v )                                   //    linear form
35 :     + on(2,u=ue) ;                                       //    boundary condition
36 :
37 : /* plot the solution */
38 : plot (Th,ps="adaptDNmix.ps");
39 : plot (u,wait=true);
40 :
41 : Vh uue = ue;
42 : real H1e = sqrt( int2d(Th) ( dx(uue)^2 + dy(uue)^2 + uue^2 ) );
43 :
44 : /* calculate the H1 Sobolev norm */
45 : Vh err0 = u0 - ue;
46 : Vh err = u - ue;
47 : Vh H1err0 = int2d(Th) ( dx(err0)^2+dy(err0)^2+err0^2 );
48 : Vh H1err = int2d(Th) ( dx(err)^2+dy(err)^2+err^2 );
49 : cout <<"Relative error in first mesh "<< int2d(Th) (H1err0)/H1e<<endl;
50 : cout <<"Relative error in adaptive mesh "<< int2d(Th) (H1err)/H1e<<endl;

```

From 24th line to 28th, adaptation of meshes are done using the base of singular term. In 42th line, $H1e=\|u_e\|_{1,\Omega}$ is calculated. In last 2 lines, the relative errors are calculated, that is,

$$\begin{aligned}\|u_h^0 - u_e\|_{1,\Omega}/H1e &= 0.120421 \\ \|u_h^a - u_e\|_{1,\Omega}/H1e &= 0.0150581\end{aligned}$$

where u_h^0 is the numerical solution in $T0h$ and u_h^a is u in this program.

9.1.7 Poisson with mixted finite element

Here we consider the Poisson equation with mixed boundary value problems: For given functions f , g_d , g_n , find p such that

$$\begin{aligned}-\Delta p &= 1 && \text{in } \Omega \\ p &= g_d && \text{on } \Gamma_D, \quad \partial p / \partial n = g_n && \text{on } \Gamma_N\end{aligned}\tag{9.17}$$

where Γ_D is a part of the boundary Γ and $\Gamma_N = \Gamma \setminus \overline{\Gamma_D}$.

The mixte formulation is: find p and \mathbf{u} such that

$$\begin{aligned}\nabla p + \mathbf{u} &= \mathbf{0} && \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= f && \text{in } \Omega \\ p &= g_d && \text{on } \Gamma_D, \quad \partial u \cdot \mathbf{n} = \mathbf{g}_n \cdot \mathbf{n} && \text{on } \Gamma_N\end{aligned}\tag{9.18}$$

where \mathbf{g}_n is a vector such that $\mathbf{g}_n \cdot \mathbf{n} = g_n$.

The variationnal formulation is,

$$\begin{aligned}\forall \mathbf{v} \in \mathbb{V}_0, \quad \int_{\Omega} p \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} &= \int_{\Gamma_d} g_d \mathbf{v} \cdot \mathbf{n} \\ \forall q \in \mathbb{P} \quad \int_{\Omega} q \nabla \cdot \mathbf{u} &= \int_{\Omega} q f \\ \partial u \cdot \mathbf{n} &= \mathbf{g}_n \cdot \mathbf{n} && \text{on } \Gamma_N\end{aligned}\tag{9.19}$$

where the functionnal space are:

$$\mathbb{P} = L^2(\Omega), \quad \mathbb{V} = H(\text{div}) = \{\mathbf{v} \in L^2(\Omega)^2, \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

and

$$\mathbb{V}_0 = \{\mathbf{v} \in \mathbb{V}; \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N\}.$$

To write, the FreeFem++ example, we have just to choose the finites elements spaces. here \mathbb{V} space is discretize with Raviart-Thomas finite element RT0 and \mathbb{P} is discretize by constant finite element P0.

Example 9.10 (LaplaceRT.edp)

```

mesh Th=square(10,10);
fespace Vh(Th,RT0);
fespace Ph(Th,P0);
func gd = 1.;
func gln = 1.;
func g2n = 1.;

Vh [u1,u2],[v1,v2];
Ph p,q;

problem laplaceMixte([u1,u2,p],[v1,v2,q],
                    solver=GMRES,eps=1.0e-10,
                    tgv=1e30,dimKrylov=150)
=
  int2d(Th) ( p*q*1e-15 // this term is here to be sur
                // that all sub matrix are inversible (LU requirement)
                + u1*v1 + u2*v2 + p*(dx(v1)+dy(v2)) + (dx(u1)+dy(u2))*q )
+ int2d(Th) ( q)
- int1d(Th,1,2,3) ( gd*(v1*N.x +v2*N.y)) // on  $\Gamma_D$ 
+ on(4,u1=gln,u2=g2n); // on  $\Gamma_N$ 

laplaceMixte;

plot ([u1,u2],coef=0.1,wait=1,ps="lapRTuv.eps",value=true);
plot (p,fill=1,wait=1,ps="laRTp.eps",value=true);

```

9.1.8 Metric Adaptation and residual error indicator

We do metric mesh adaption and compute the classical residual error indicator η_T on the element T for the Poisson problem.

Example 9.11 (adaptindicatorP2.edp) *First, we solve the same problem as in a previous example.*

```

1 : border ba(t=0,1.0){x=t; y=0; label=1;}; // see Fig,5.15
2 : border bb(t=0,0.5){x=1; y=t; label=2;};
3 : border bc(t=0,0.5){x=1-t; y=0.5;label=3;};
4 : border bd(t=0.5,1){x=0.5; y=t; label=4;};
5 : border be(t=0.5,1){x=1-t; y=1; label=5;};
6 : border bf(t=0.0,1){x=0; y=1-t;label=6;};

```

```

7 : mesh Th = buildmesh (ba(6) + bb(4) + bc(4) +bd(4) + be(4) + bf(6));
8 : savemesh(Th,"th.msh");
9 : fespace Vh(Th,P2);
10 : fespace Nh(Th,P0);
11 : Vh u,v;
12 : Nh rho;
13 : real[int] viso(21);
14 : for (int i=0;i<viso.n;i++)
15 :   viso[i]=10.^(+(i-16.)/2.);
16 : real error=0.01;
17 : func f=(x-y);
18 : problem Problm1(u,v,solver=CG,eps=1.0e-6) =
19 :   int2d(Th,qforder=5) ( u*v*1.0e-10+ dx(u)*dx(v) + dy(u)*dy(v) )
20 :   + int2d(Th,qforder=5) ( -f*v );
21 : /*****

```

Now, the local error indicator η_T is:

$$\eta_T = \left(h_T^2 \|f + \Delta u_h\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_K} h_e \left\| \left[\frac{\partial u_h}{\partial n_k} \right] \right\|_{L^2(e)}^2 \right)^{\frac{1}{2}}$$

where h_T is the longest's edge of T , \mathcal{E}_T is the set of T edge not on $\Gamma = \partial\Omega$, n_T is the outside unit normal to K , h_e is the length of edge e , $[g]$ is the jump of the function g across edge (left value minus right value).

Of course, we can use a variational form to compute η_T^2 , with test function constant function in each triangle.

```

29 : *****/
30 :
31 : varf indicator2(uu,chiK) =
32 :   intalledges(Th) (chiK*lenEdge*square(jump(N.x*dx(u)+N.y*dy(u))))
33 :   +int2d(Th) (chiK*square(hTriangle*(f+dxx(u)+dyy(u))) );
34 : for (int i=0;i< 4;i++)
35 : {
36 :   Problm1;
37 :   cout << u[].min << " " << u[].max << endl;
38 :   plot(u,wait=1);
39 :   cout << " indicator2 " << endl;
40 :
41 :   rho[] = indicator2(0,Nh);
42 :   rho=sqrt(rho);
43 :   cout << "rho =   min " << rho[].min << " max=" << rho[].max << endl;
44 :   plot(rho,fill=1,wait=1,cmm="indicator density ",ps="rhoP2.eps",
         value=1,viso=viso,nbiso=viso.n);
45 :   plot(Th,wait=1,cmm="Mesh ",ps="ThrhoP2.eps");
46 :   Th=adaptmesh(Th, [dx(u),dy(u)],err=error,anisomax=1);
47 :   plot(Th,wait=1);
48 :   u=u;
49 :   rho=rho;
50 :   error = error/2;
51 : } ;

```

If the method is correct, we expect to look the graphics by an almost constant function η on your computer as in Fig. 9.11.

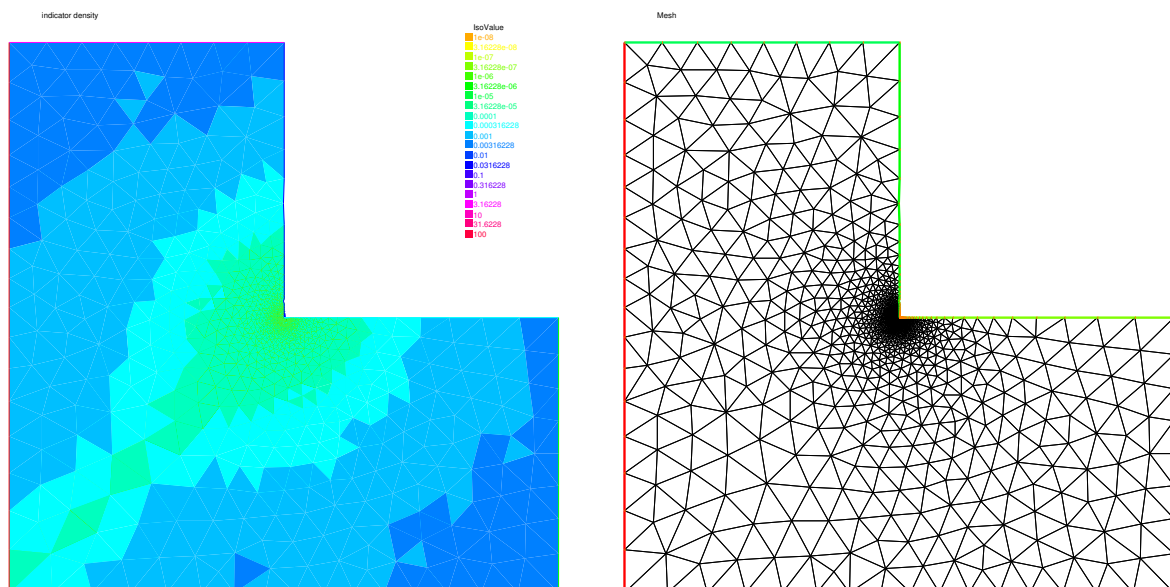


Figure 9.11: Density of the error indicator with isotropic P_2 metric

9.1.9 Adaptation using residual error indicator

In the previous example we compute the error indicator, now we use it, to adapt the mesh. The new mesh size is given by the following formulae:

$$h_{n+1}(x) = \frac{h_n(x)}{f_n(\eta_K(x))}$$

where $\eta_n(x)$ is the level of error at point x given by the local error indicator, h_n is the previous “mesh size” field, and f_n is a user function define by $f_n = \min(3, \max(1/3, \eta_n/\eta_n^*))$ where $\eta_n^* = \text{mean}(\eta_n)c$, and c is an user coefficient generally close to one.

Example 9.12 (AdaptResidualErrorIndicator.edp)

First a macro MeshSizecomputation to get a P_1 mesh size as the average of edge length.

```
// macro the get the current mesh size
// parameter
// in: Th the mesh
// Vh P1 fespace on Th
// out :
// h: the Vh finite element finite set to the current mesh size
macro MeshSizecomputation(Th,Vh,h)
{ /* Th mesh Vh P1 finite element space
  h the P1 mesh size value */
  real[int] count(Th.nv);
  /* mesh size (lenEdge = integral(e) 1 ds) */
  varf vmeshsize(u,v)=intalldges(Th,qfnbpE=1)(v);
  /* number of edge / par vertex */
  varf vedgecount(u,v)=intalldges(Th,qfnbpE=1)(v/lenEdge);
  /*
  computation of the mesh size
  ----- */
```

```

count=vedgecount(0,Vh);
h[]=0.;
h[]=vmeshsizen(0,Vh);
cout << " count min = "<< count.min << " " << count.max << endl;
h[]=h[]./count;
    cout << " -- bound meshsize = " <<h[].min << " " << h[].max << endl;
} //    end of macro MeshSizecomputation

```

A second macro to remesh according to the new mesh size.

```

//    macro to remesh according the de residual indicator
//    in:
//    Th the mesh
//    Ph P0 fespace on Th
//    Vh P1 fespace on Th
//    vindicator the varf of to evaluate the indicator to 2
//    coef on etameam ..
//    -----

macro ReMeshIndicator(Th,Ph,Vh,vindicator,coef)
{
Vh h=0;
/*evalutate the mesh size */
MeshSizecomputation(Th,Vh,h);
Ph etak;
etak[]=vindicator(0,Ph);
etak[]=sqrt(etak[]);
real etastar= coef*(etak[].sum/etak[].n);
cout << " etastar = " << etastar << " sum=" << etak[].sum << " " << endl;

/* here etaK is discontinous
   we use the P1 L2 projection with mass lumping . */

Vh fn,sigma;
varf veta(unused,v)=int2d(Th)(etak*v);
varf vun(unused,v)=int2d(Th)(1*v);
fn[] = veta(0,Vh);
sigma[]= vun(0,Vh);
fn[]= fn[]./ sigma[];
fn = max(min(fn/etastar,3.),0.3333) ;

/* new mesh size */
h = h / fn ;
/* plot(h,wait=1); */
/* build the new mesh */
Th=adaptmesh(Th,IsMetric=1,h,splitpbedge=1,nbvx=10000);
}

```

We skip the mesh construction, see the previous example,

```

//    FE space definition ---
fespace Vh(Th,P1);           //    for the mesh size and solution
fespace Ph(Th,P0);           //    for the error indicator

real hinit=0.2;               //    initial mesh size
Vh h=hinit;                   //    the FE function for the mesh size
//    to build a mesh with a given mesh size : meshsize

```

```

Th=adaptmesh(Th,h,IsMetric=1,splitpbedge=1,nbvx=10000);
plot(Th,wait=1,ps="RRI-Th-init.eps");
Vh u,v;

func f=(x-y);

problem Poisson(u,v) =
    int2d(Th,qforder=5) ( u*v*1.0e-10+ dx(u)*dx(v) + dy(u)*dy(v) )
    - int2d(Th,qforder=5) ( f*v );

varf indicator2(unused,chiK) =
    intalledges(Th) (chiK*lenEdge*square(jump(N.x*dx(u)+N.y*dy(u))))
    +int2d(Th) (chiK*square(hTriangle*(f+dxx(u)+dyy(u))) );

for (int i=0;i< 10;i++)
{
u=u;
Poisson;
plot(Th,u,wait=1);
real cc=0.8;
if(i>5) cc=1;
ReMeshIndicator(Th,Ph,Vh,indicator2,cc);
plot(Th,wait=1);
}

```

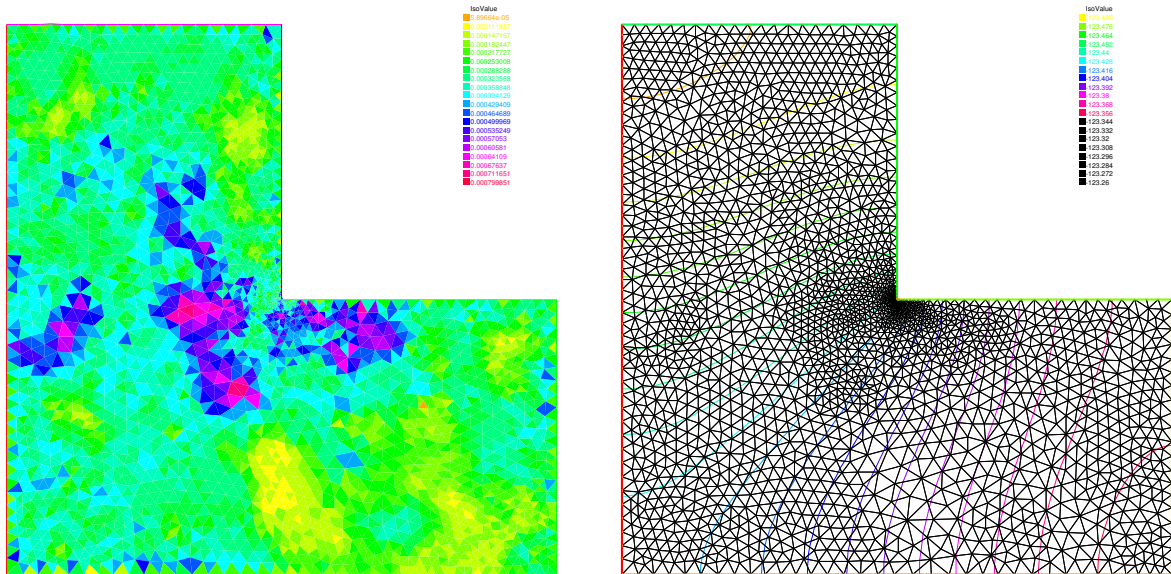


Figure 9.12: the error indicator with isotropic P_1 , the mesh and isovalue of the solution

9.2 Elasticity

Consider an elastic plate with undeformed shape $\Omega \times]-h, h[$ in \mathbb{R}^3 , $\Omega \subset \mathbb{R}^2$. By the deformation of the plate, we assume that a point $P(x_1, x_2, x_3)$ moves to $\mathcal{P}(\xi_1, \xi_2, \xi_3)$. The vector

$\mathbf{u} = (u_1, u_2, u_3) = (\xi_1 - x_1, \xi_2 - x_2, \xi_3 - x_3)$ is called the *displacement vector*. By the deformation, the line segment $\overline{\mathbf{x}, \mathbf{x} + \tau \Delta \mathbf{x}}$ moves approximately to $\overline{\mathbf{x} + u(\mathbf{x}), \mathbf{x} + \tau \Delta \mathbf{x} + u(\mathbf{x} + \tau \Delta \mathbf{x})}$ for small τ , where $\mathbf{x} = (x_1, x_2, x_3)$, $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \Delta x_3)$. We now calculate the ratio between two segments

$$\eta(\tau) = \tau^{-1} |\Delta \mathbf{x}|^{-1} (|u(\mathbf{x} + \tau \Delta \mathbf{x}) - u(\mathbf{x}) + \tau \Delta \mathbf{x}| - \tau |\Delta \mathbf{x}|)$$

then we have (see e.g. [16, p.32])

$$\lim_{\tau \rightarrow 0} \eta(\tau) = (1 + 2e_{ij}\nu_i\nu_j)^{1/2} - 1, \quad 2e_{ij} = \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where $\nu_i = \Delta x_i |\Delta \mathbf{x}|^{-1}$. If the deformation is *small*, then we may consider that

$$(\partial u_k / \partial x_i)(\partial u_k / \partial x_i) \approx 0$$

and the following is called *small strain tensor*

$$\varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

The tensor e_{ij} is called *finite strain tensor*.

Consider the small plane $\Delta \Pi(\mathbf{x})$ centered at \mathbf{x} with the unit normal direction $\mathbf{n} = (n_1, n_2, n_3)$, then the surface on $\Delta \Pi(\mathbf{x})$ at \mathbf{x} is

$$(\sigma_{1j}(\mathbf{x})n_j, \sigma_{2j}(\mathbf{x})n_j, \sigma_{3j}(\mathbf{x})n_j)$$

where $\sigma_{ij}(\mathbf{x})$ is called *stress tensor* at \mathbf{x} . Hooke's law is the assumption of a linear relation between σ_{ij} and ε_{ij} such as

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}(\mathbf{x})\varepsilon_{ij}(\mathbf{x})$$

with the symmetry $c_{ijkl} = c_{jikl}, c_{ijkl} = c_{ijlk}, c_{ijkl} = c_{klij}$.

If Hooke's tensor $c_{ijkl}(\mathbf{x})$ do not depend on the choice of coordinate system, the material is called *isotropic* at \mathbf{x} . If c_{ijkl} is constant, the material is called *homogeneous*. In homogeneous isotropic case, there is *Lamé constants* λ, μ (see e.g. [16, p.43]) satisfying

$$\sigma_{ij} = \lambda \delta_{ij} \text{div} u + 2\mu \varepsilon_{ij} \quad (9.20)$$

where δ_{ij} is Kronecker's delta. We assume that the elastic plate is fixed on $\Gamma_D \times]-h, h[$, $\Gamma_D \subset \partial \Omega$. If the body force $\mathbf{f} = (f_1, f_2, f_3)$ is given in $\Omega \times]-h, h[$ and surface force g is given in $\Gamma_N \times]-h, h[$, $\Gamma_N = \partial \Omega \setminus \Gamma_D$, then the equation of equilibrium is given as follows:

$$-\partial_j \sigma_{ij} = f_i \text{ in } \Omega \times]-h, h[, \quad i = 1, 2, 3 \quad (9.21)$$

$$\sigma_{ij} n_j = g_i \text{ on } \Gamma_N \times]-h, h[, \quad u_i = 0 \text{ on } \Gamma_D \times]-h, h[, \quad i = 1, 2, 3 \quad (9.22)$$

We now explain the plain elasticity.

Plain strain: On the end of plate, the contact condition $u_3 = 0$, $g_3 = 0$ is satisfied. In this case, we can suppose that $f_3 = g_3 = u_3 = 0$ and $\mathbf{u}(x_1, x_2, x_3) = \bar{\mathbf{u}}(x_1, x_2)$ for all $-h < x_3 < h$.

Plain stress: The cylinder is assumed to be very thin and subjected to no load on the ends $x_3 = \pm h$, that is,

$$\sigma_{3i} = 0, \quad x_3 = \pm h, \quad i = 1, 2, 3$$

The assumption leads that $\sigma_{3i} = 0$ in $\Omega \times]-h, h[$ and $\mathbf{u}(x_1, x_2, x_3) = \bar{\mathbf{u}}(x_1, x_2)$ for all $-h < x_3 < h$.

Generalized plain stress: The cylinder is subjected to no load at $x_3 = \pm h$. Introducing the mean values with respect to thickness,

$$\bar{u}_i(x_1, x_2) = \frac{1}{2h} \int_{-h}^h u(x_1, x_2, x_3) dx_3$$

and we derive $\bar{u}_3 \equiv 0$. Similarly we define the mean values \bar{f}, \bar{g} of the body force and surface force as well as the mean values $\bar{\varepsilon}_{ij}$ and $\bar{\sigma}_{ij}$ of the components of stress and strain, respectively.

In what follows we omit the overlines of $\bar{u}, \bar{f}, \bar{g}, \bar{\varepsilon}_{ij}$ and $\bar{\sigma}_{ij}$. Then we obtain similar equation of equilibrium given in (9.21) replacing $\Omega \times]-h, h[$ with Ω and changing $i = 1, 2$. In the case of plane stress, $\sigma_{ij} = \lambda^* \delta_{ij} \operatorname{div} u + 2\mu \varepsilon_{ij}$, $\lambda^* = (2\lambda\mu)/(\lambda + \mu)$.

The equations of elasticity are naturally written in variational form for the displacement vector $u(x) \in V$ as

$$\int_{\Omega} [2\mu \varepsilon_{ij}(u) \varepsilon_{ij}(v) + \lambda \varepsilon_{ii}(u) \varepsilon_{jj}(v)] = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma} \mathbf{g} \cdot \mathbf{v}, \forall v \in V$$

where V is the linear closed subspace of $H^1(\Omega)^2$.

Example 9.13 (Beam.edp) Consider elastic plate with the undeformed rectangle shape $]0, 10[\times]0, 2[$. The body force is the gravity force \mathbf{f} and the boundary force \mathbf{g} is zero on lower and upper side. On the two vertical sides of the beam are fixed.

```

//      a weighting beam sitting on a

int bottombeam = 2;
border a(t=2,0) { x=0; y=t ;label=1;}; //      left beam
border b(t=0,10) { x=t; y=0 ;label=bottombeam;}; //      bottom of beam
border c(t=0,2) { x=10; y=t ;label=1;}; //      righth beam
border d(t=0,10) { x=10-t; y=2; label=3;}; //      top beam
real E = 21.5;
real sigma = 0.29;
real mu = E/(2*(1+sigma));
real lambda = E*sigma/((1+sigma)*(1-2*sigma));
real gravity = -0.05;
mesh th = buildmesh( b(20)+c(5)+d(20)+a(5));
fespace Vh(th, [P1,P1]);
Vh [uu,vv], [w,s];
cout << "lambda,mu,gravity =" << lambda << " " << mu << " " << gravity << endl;
//      deformation of a beam under its own weight
//      see lame.edp example 3.8
real sqrt2=sqrt(2.); //      see lame.edp example 3.8
macro epsilon(u1,u2) [dx(u1),dy(u2),(dy(u1)+dx(u2))/sqrt2] //      EOM
macro div(u,v) ( dx(u)+dy(v) ) //      EOM

solve bb([uu,vv],[w,s])=
  int2d(th) (
    lambda*div(w,s)*div(uu,vv)
    +2.*mu*( epsilon(w,s)'*epsilon(uu,vv) )
  )
+ int2d(th) (-gravity*s)
+ on(1,uu=0,vv=0)
;

plot([uu,vv],wait=1);

```



```

plot([uu,vv],wait=1,bb=[[-0.5,2.5],[2.5,-0.5]]);
mesh th1 = movemesh(th, [x+uu, y+vv]);
plot(th1,wait=1);

```

Example 9.14 (beam-3d.edp) Consider elastic box with the undeformed parallelepiped shape $]0, 5[\times]0, 1[\times]0, 1[$. The body force is the gravity force \mathbf{f} and the boundary force \mathbf{g} is zero on all face except one the one vertical left face where the beam is fixed.

```

include "cube.idp"
int[int] Nxyz=[20,5,5];
real [int,int] Bxyz=[[0.,5.],[0.,1.],[0.,1.]];
int [int,int] Lxyz=[[1,2],[2,2],[2,2]];
mesh3 Th=Cube(Nxyz,Bxyz,Lxyz);

real E = 21.5e4, sigma = 0.29;
real mu = E/(2*(1+sigma));
real lambda = E*sigma/((1+sigma)*(1-2*sigma));
real gravity = -0.05;

fespace Vh(Th, [P1,P1,P1]);
Vh [u1,u2,u3], [v1,v2,v3];
cout << "lambda,mu,gravity ="<<lambda<< " " << mu << " " << gravity << endl;

real sqrt2=sqrt(2.);
macro epsilon(u1,u2,u3) [dx(u1),dy(u2),dz(u3), (dz(u2)+dy(u3))/sqrt2,
                        (dz(u1)+dx(u3))/sqrt2, (dy(u1)+dx(u2))/sqrt2] // EOM
macro div(u1,u2,u3) ( dx(u1)+dy(u2)+dz(u3) ) // EOM

solve Lamé([u1,u2,u3],[v1,v2,v3])=
  int3d(Th) (
    lambda*div(u1,u2,u3)*div(v1,v2,v3)
    +2.*mu*( epsilon(u1,u2,u3)'*epsilon(v1,v2,v3) ) // '
  )
  - int3d(Th) (gravity*v3)
  + on(1,u1=0,u2=0,u3=0)
  ;
real dmax= u1[].max;
cout << " max displacement = " << dmax << endl;
real coef= 0.1/dmax;
int[int] ref2=[1,0,2,0];
mesh3 Thm=movemesh3(Th,transfo=[x+u1*coef,y+u2*coef,z+u3*coef],label=ref2);
Thm=change(Thm,label=ref2);
plot(Th,Thm, wait=1,cmm="coef amplification = "+coef ); // see fig ??

```

9.2.1 Fracture Mechanics

Consider the plate with the crack whose undeformed shape is a curve Σ with the two edges γ_1, γ_2 . We assume the stress tensor σ_{ij} is the state of plate stress regarding $(x, y) \in \Omega_\Sigma = \Omega \setminus \Sigma$. Here Ω stands for the undeformed shape of elastic plate without crack. If the part Γ_N of the boundary $\partial\Omega$ is fixed and a load $\mathcal{L} = (\mathbf{f}, \mathbf{g}) \in L^2(\Omega)^2 \times L^2(\Gamma_N)^2$ is given, then the displacement \mathbf{u} is the minimizer of the potential energy functional

$$\mathcal{E}(\mathbf{v}; \mathcal{L}, \Omega_\Sigma) = \int_{\Omega_\Sigma} \{w(x, \mathbf{v}) - \mathbf{f} \cdot \mathbf{v}\} - \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}$$

over the functional space $V(\Omega_\Sigma)$,

$$V(\Omega_\Sigma) = \{\mathbf{v} \in H^1(\Omega_\Sigma)^2; \mathbf{v} = 0 \text{ on } \Gamma_D = \partial\Omega \setminus \overline{\Gamma_N}\},$$

where $w(x, \mathbf{v}) = \sigma_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})/2$,

$$\sigma_{ij}(\mathbf{v}) = C_{ijkl}(x)\varepsilon_{kl}(\mathbf{v}), \quad \varepsilon_{ij}(\mathbf{v}) = (\partial v_i / \partial x_j + \partial v_j / \partial x_i) / 2, \quad (C_{ijkl} : \text{Hooke's tensor}).$$

If the elasticity is homogeneous isotropic, then the displacement $\mathbf{u}(x)$ is decomposed in an open neighborhood U_k of γ_k as in (see e.g. [17])

$$\mathbf{u}(x) = \sum_{l=1}^2 K_l(\gamma_k) r_k^{1/2} S_{kl}^C(\theta_k) + \mathbf{u}_{k,R}(x) \quad \text{for } x \in \Omega_\Sigma \cap U_k, \quad k = 1, 2 \quad (9.23)$$

with $\mathbf{u}_{k,R} \in H^2(\Omega_\Sigma \cap U_k)^2$, where $U_k, k = 1, 2$ are open neighborhoods of γ_k such that $\partial L_1 \cap U_1 = \gamma_1$, $\partial L_m \cap U_2 = \gamma_2$, and

$$\begin{aligned} S_{k1}^C(\theta_k) &= \frac{1}{4\mu} \frac{1}{(2\pi)^{1/2}} \begin{bmatrix} [2\kappa - 1] \cos(\theta_k/2) - \cos(3\theta_k/2) \\ -[2\kappa + 1] \sin(\theta_k/2) + \sin(3\theta_k/2) \end{bmatrix}, \\ S_{k2}^C(\theta_k) &= \frac{1}{4\mu} \frac{1}{(2\pi)^{1/2}} \begin{bmatrix} -[2\kappa - 1] \sin(\theta_k/2) + 3 \sin(3\theta_k/2) \\ -[2\kappa + 1] \cos(\theta_k/2) + \cos(3\theta_k/2) \end{bmatrix}. \end{aligned} \quad (9.24)$$

where μ is the shear modulus of elasticity, $\kappa = 3 - 4\nu$ (ν is the Poisson's ratio) for plane strain and $\kappa = \frac{3-\nu}{1+\nu}$ for plane stress.

The coefficients $K_1(\gamma_i)$ and $K_2(\gamma_i)$, which are important parameters in fracture mechanics, are called stress intensity factors of the opening mode (mode I) and the sliding mode (mode II), respectively.

For simplicity, we consider the following simple crack

$$\Omega = \{(x, y) : -1 < x < 1, -1 < y < 1\}, \quad \Sigma = \{(x, y) : -1 \leq x \leq 0, y = 0\}$$

with only one crack tip $\gamma = (0, 0)$. Unfortunately, FreeFem++ cannot treat crack, so we use the modification of the domain with U-shape channel (see Fig. 5.30) with $d = 0.0001$. The undeformed crack Σ is approximated by

$$\begin{aligned} \Sigma_d &= \{(x, y) : -1 \leq x \leq -10 * d, -d \leq y \leq d\} \\ &\cup \{(x, y) : -10 * d \leq x \leq 0, -d + 0.1 * x \leq y \leq d - 0.1 * x\} \end{aligned}$$

and $\Gamma_D = \mathbb{R}$ in Fig. 5.30. In this example, we use three technique:

- Fast Finite Element Interpolator from the mesh Th to Zoom for the scale-up of near γ .

- After obtaining the displacement vector $\mathbf{u} = (u, v)$, we shall watch the deformation of the crack near γ as follows,

```
mesh Plate = movemesh(Zoom, [x+u, y+v]);
plot(Plate);
```

- Adaptivity is an important technique here, because a large singularity occurs at γ as shown in (9.23).

The first example creates mode I deformation by the opposed surface force on B and T in the vertical direction of Σ , and the displacement is fixed on R .

In a laboratory, fracture engineers use photoelasticity to make stress field visible, which shows the principal stress difference

$$\sigma_1 - \sigma_2 = \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} \quad (9.25)$$

where σ_1 and σ_2 are the principal stresses. In opening mode, the photoelasticity make symmetric pattern concentrated at γ .

Example 9.15 (Crack Opening, $K_2(\gamma) = 0$) {CrackOpen.edp}

```
real d = 0.0001;
int n = 5;
real cb=1, ca=1, tip=0.0;
border L1(t=0,ca-d) { x=-cb; y=-d-t; }
border L2(t=0,ca-d) { x=-cb; y=ca-t; }
border B(t=0,2) { x=cb*(t-1); y=-ca; }
border C1(t=0,1) { x=-ca*(1-t)+(tip-10*d)*t; y=d; }
border C21(t=0,1) { x=(tip-10*d)*(1-t)+tip*t; y=d*(1-t); }
border C22(t=0,1) { x=(tip-10*d)*t+tip*(1-t); y=-d*t; }
border C3(t=0,1) { x=(tip-10*d)*(1-t)-ca*t; y=-d; }
border C4(t=0,2*d) { x=-ca; y=-d+t; }
border R(t=0,2) { x=cb; y=cb*(t-1); }
border T(t=0,2) { x=cb*(1-t); y=ca; }
mesh Th = buildmesh (L1 (n/2)+L2 (n/2)+B (n)
                    +C1 (n)+C21 (3)+C22 (3)+C3 (n)+R (n)+T (n));

cb=0.1; ca=0.1;
plot(Th,wait=1);
mesh Zoom = buildmesh (L1 (n/2)+L2 (n/2)+B (n)+C1 (n)
                    +C21 (3)+C22 (3)+C3 (n)+R (n)+T (n));

plot(Zoom,wait=1);
real E = 21.5;
real sigma = 0.29;
real mu = E/(2*(1+sigma));
real lambda = E*sigma/((1+sigma)*(1-2*sigma));
fespace Vh(Th, [P2,P2]);
fespace zVh(Zoom,P2);
Vh [u,v], [w,s];
solve Problem([u,v],[w,s]) =
    int2d(Th) (
        2*mu*(dx(u)*dx(w) + ((dx(v)+dy(u))*(dx(s)+dy(w)))/4 )
        + lambda*(dx(u)+dy(v))*(dx(w)+dy(s))/2
    )
    -int1d(Th,T) (0.1*(4-x)*s) +int1d(Th,B) (0.1*(4-x)*s)
    +on(R,u=0)+on(R,v=0);
```

// fixed

```

zVh Sx, Sy, Sxy, N;
for (int i=1; i<=5; i++)
{
  mesh Plate = movemesh(Zoom, [x+u, y+v]);           // deformation near  $\gamma$ 
  Sx = lambda*(dx(u)+dy(v)) + 2*mu*dx(u);
  Sy = lambda*(dx(u)+dy(v)) + 2*mu*dy(v);
  Sxy = mu*(dy(u) + dx(v));
  N = 0.1*1*sqrt((Sx-Sy)^2+4*Sxy^2);                 // principal stress difference
  if (i==1) {
    plot(Plate, ps="1stCOD.eps", bw=1);               // Fig. 9.13
    plot(N, ps="1stPhoto.eps", bw=1);                 // Fig. 9.13
  } else if (i==5) {
    plot(Plate, ps="LastCOD.eps", bw=1);              // Fig. 9.14
    plot(N, ps="LastPhoto.eps", bw=1);                // Fig. 9.14
    break;
  }
  Th=adaptmesh(Th, [u, v]);
  Problem;
}

```

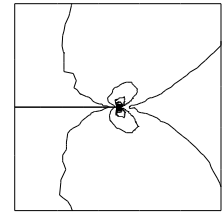
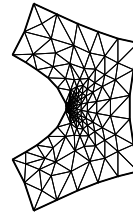
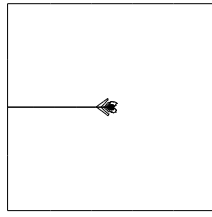
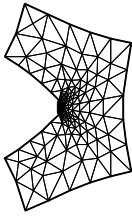


Figure 9.13: Crack open displacement (COD) and Principal stress difference in the first mesh

Figure 9.14: COD and Principal stress difference in the last adaptive mesh

It is difficult to create mode II deformation by the opposed shear force on B and T that is observed in a laboratory. So we use the body shear force along Σ , that is, the x -component f_1 of the body force \mathbf{f} is given by

$$f_1(x, y) = H(y - 0.001) * H(0.1 - y) - H(-y - 0.001) * H(y + 0.1)$$

where $H(t) = 1$ if $t > 0$; $= 0$ if $t < 0$.

Example 9.16 (Crack Sliding, $K_2(\gamma) = 0$) (use the same mesh Th)

```

cb=0.01; ca=0.01;
mesh Zoom = buildmesh (L1(n/2)+L2(n/2)+B(n)+C1(n)
                        +C21(3)+C22(3)+C3(n)+R(n)+T(n));
(use same FE-space Vh and elastic modulus)
fespace Vh1(Th, P1);
Vh1 fx = ((y>0.001)*(y<0.1))-((y<-0.001)*(y>-0.1));

solve Problem([u, v], [w, s]) =

```

```

int2d(Th) (
    2*mu*(dx(u)*dx(w) + ((dx(v)+dy(u))*(dx(s)+dy(w)))/4 )
    + lambda*(dx(u)+dy(v))*(dx(w)+dy(s))/2
)
-int2d(Th) (fx*w)
+on(R,u=0)+on(R,v=0); // fixed
;

for (int i=1; i<=3; i++)
{
    mesh Plate = movemesh(Zoom,[x+u,y+v]); // deformation near γ
    Sx = lambda*(dx(u)+dy(v)) + 2*mu*dx(u);
    Sy = lambda*(dx(u)+dy(v)) + 2*mu*dy(v);
    Sxy = mu*(dy(u) + dx(v));
    N = 0.1*sqrt((Sx-Sy)^2+4*Sxy^2); // principal stress difference
    if (i==1) {
        plot(Plate,ps="1stCOD2.eps",bw=1); // Fig. 9.16
        plot(N,ps="1stPhoto2.eps",bw=1); // Fig. 9.15
    } else if (i==3) {
        plot(Plate,ps="LastCOD2.eps",bw=1); // Fig. 9.16
        plot(N,ps="LastPhoto2.eps",bw=1); // Fig. 9.16
        break;
    }
    Th=adaptmesh(Th,[u,v]);
    Problem;
}

```

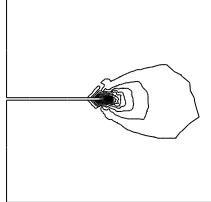
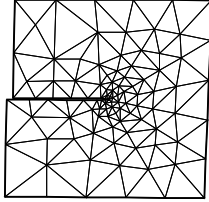


Figure 9.15: (COD) and Principal stress difference in the first mesh

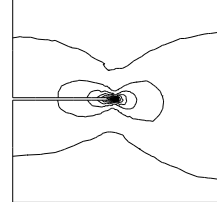
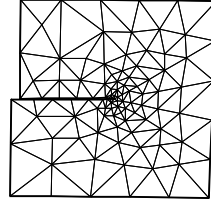


Figure 9.16: COD and Principal stress difference in the last adaptive mesh

9.3 Nonlinear Static Problems

Here we propose to solve the following non-linear academic problem of minimization of a functional

$$J(u) = \int_{\Omega} \frac{1}{2} f(|\nabla u|^2) - u * b$$

where u is function of $H_0^1(\Omega)$ and f defined by

$$f(x) = a * x + x - \ln(1 + x), \quad f'(x) = a + \frac{x}{1 + x}, \quad f''(x) = \frac{1}{(1 + x)^2}$$

9.3.1 Newton-Raphson algorithm

Now, we solve the Euler problem $\nabla J(u) = 0$ with Newton-Raphson algorithm, that is,

$$u^{n+1} = u^n - (\nabla^2 J(u^n))^{-1} * \nabla J(u^n)$$

First we introduce the two variational form `vdJ` and `vhJ` to compute respectively ∇J and $\nabla^2 J$

```
//      method of Newton-Raphson to solve dJ(u)=0;                                     //
```

$$u^{n+1} = u^n - \left(\frac{\partial dJ}{\partial u_i} \right)^{-1} * dJ(u^n)$$

```
//      -----
Ph dalpha ;                                     //      to store 2f''(|\nabla u|^2) optimisation

//      the variational form of evaluate dJ = \nabla J
//      -----
//      dJ = f'()*( dx(u)*dx(vh) + dy(u)*dy(vh) )
varf vdJ(uh,vh) = int2d(Th) ( alpha*( dx(u)*dx(vh) + dy(u)*dy(vh) ) - b*vh )
+ on(1,2,3,4, uh=0);

//      the variational form of evaluate ddJ = \nabla^2 J
//      hJ(uh,vh) = f'()*( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
//                  + 2*f''() ( dx(u)*dx(uh) + dy(u)*dy(uh) ) * (dx(u)*dx(vh) +
dy(u)*dy(vh))
varf vhJ(uh,vh) = int2d(Th) ( alpha*( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ dalpha*( dx(u)*dx(vh) + dy(u)*dy(vh) )*( dx(u)*dx(uh) + dy(u)*dy(uh) ) )
+ on(1,2,3,4, uh=0);

//      the Newton algorithm
Vh v,w;
u=0;
for (int i=0;i<100;i++)
{
  alpha =      df( dx(u)*dx(u) + dy(u)*dy(u) ) ;           //      optimization
  dalpha = 2*ddf( dx(u)*dx(u) + dy(u)*dy(u) ) ;           //      optimization
  v[] = vdJ(0,Vh);                                           //      v = \nabla J(u)
  real res= v[]'*v[];                                         //      the dot product
  cout << i << " residu^2= " << res << endl;
  if ( res< 1e-12) break;
  matrix H= vhJ(Vh,Vh,factorize=1,solver=LU);               //
```

$$w[] = H^{-1} * v[];$$

$$u[] -= w[];$$

```
}
plot (u,wait=1,cmm="solution with Newton-Raphson");
```

Remark: This example is in `Newton.edp` file of `examples++-tutorial` directory.

9.4 Eigenvalue Problems

This section depends on your installation of FreeFem++ ; you need to have compiled (see README_arpack), ARPACK. This tools is available in FreeFem++ if the word “eigenvalue” appear in line “Load:”, like:

```
-- FreeFem++ v1.28 (date Thu Dec 26 10:56:34 CET 2002)
file : LapEigenValue.edp
Load: lg_fem lg_mesh eigenvalue
```

This tools is based on the arpack++¹ the object-oriented version of ARPACK eigenvalue package [1].

The function EigenValue computes the generalized eigenvalue of $Au = \lambda Bu$ where $\sigma = \sigma$ is the shift of the method. The matrix OP is defined with $A - \sigma B$. The return value is the number of converged eigenvalue (can be greater than the number of eigen value nev=)

```
int k=EigenValue(OP,B,nev= , sigma= );
```

where the matrix $OP = A - \sigma B$ with a solver and boundary condition, and the matrix B . From version 3.31 you have also a functional interface like

```
int k=EigenValue(n,FOP1,FB,nev= , sigma= );
```

where n is the size of the matrix and the now the matrix defined throught function which defined the respectively the matrix product of OP^{-1} and B , like in

```
int n = OP1.n;
func real[int] FOP1(real[int] & u) { real[int] Au=OP^-1*u;return Au;}
func real[int] FB(real[int] & u) { real[int] Au=B*u;return Au;}
```

Note 9.1 Boundary condition and Eigenvalue Problems

The locking (Dirichlet) boundary condition is make with exact penalization so we put $1e30=tgv$ on the diagonal term of the locked degree of freedom (see equation (6.31)). So take Dirichlet boundary condition just on A and not on B . because we solve $w = OP^{-1} * B * v$.

If you put locking (Dirichlet) boundary condition on B matrix (with key work **on**) you get small spurious modes (10^{-30}), due to boundary condition, but if you forget the locking boundary condition on B matrix (no key work "on") you get huge spurious (10^{30}) modes associated to these boundary conditons. We compute only small mode, so we get the good one in this case.

sym= the problem is symmetric (all the eigen value are real)

nev= the number desired eigenvalues (nev) close to the shift.

value= the array to store the real part of the eigenvalues

ivalue= the array to store the imag. part of the eigenvalues

vector= the FE function array to store the eigenvectors

¹<http://www.caam.rice.edu/software/ARPACK/>

rawvector= an array of type `real[int,int]` to store eigenvectors by column. (up to version 2-17).

For real non symmetric problems, complex eigenvectors are given as two consecutive vectors, so if eigenvalue k and $k+1$ are complex conjugate eigenvalues, the k th vector will contain the real part and the $k+1$ th vector the imaginary part of the corresponding complex conjugate eigenvectors.

tol= the relative accuracy to which eigenvalues are to be determined;

sigma= the shift value;

maxit= the maximum number of iterations allowed;

ncv= the number of Arnoldi vectors generated at each iteration of ARPACK.

Example 9.17 (lapEigenValue.edp) *In the first example, we compute the eigenvalues and the eigenvectors of the Dirichlet problem on square $\Omega =]0, \pi[^2$.*

The problem is to find: λ , and ∇u_λ in $\mathbb{R} \times H_0^1(\Omega)$

$$\int_{\Omega} \nabla u_\lambda \nabla v = \lambda \int_{\Omega} uv \quad \forall v \in H_0^1(\Omega)$$

The exact eigenvalues are $\lambda_{n,m} = (n^2 + m^2)$, $(n, m) \in \mathbb{N}_^2$ with the associated eigenvectors are $u_{m,n} = \sin(nx) * \sin(my)$.*

We use the generalized inverse shift mode of the `arpack++` library, to find 20 eigenvalues and eigenvectors close to the shift value $\sigma = 20$.

```
//      Computation of the eigen value and eigen vector of the
//      Dirichlet problem on square ]0,pi[^2
//      -----
//      we use the inverse shift mode
//      the shift is given with the real sigma
//      -----
//      find  $\lambda$  and  $u_\lambda \in H_0^1(\Omega)$  such that:
//      
$$\int_{\Omega} \nabla u_\lambda \nabla v = \lambda \int_{\Omega} u_\lambda v, \forall v \in H_0^1(\Omega)$$

verbosity=10;
mesh Th=square(20,20,[pi*x,pi*y]);
fespace Vh(Th,P2);
Vh u1,u2;

real sigma = 20; //      value of the shift

//      OP = A - sigma B ; // the shifted matrix
varf op(u1,u2)= int2d(Th)( dx(u1)*dx(u2) + dy(u1)*dy(u2) - sigma*u1*u2 )
//      Boundary condition
+ on(1,2,3,4,u1=0) ;

varf b([u1],[u2]) = int2d(Th)( u1*u2 ); //      no Boundary condition see note
9.1
matrix OP= op(Vh,Vh,solver=Crout,factorize=1); //      crout solver because the
matrix in not positive
matrix B= b(Vh,Vh,solver=CG,eps=1e-20);

//      important remark:
```



```

//      the boundary condition is make with exact penalization:
//      we put 1e30=tdv on the diagonal term of the lock degree of freedom.
//      So take Dirichlet boundary condition just on a variational form
//      and not on b variational form.
//      because we solve  $w = OP^{-1} * B * v$ 

int nev=20; //      number of computed eigen value close to sigma

real[int] ev(nev); //      to store the nev eigenvalue
Vh[int] eV(nev); //      to store the nev eigenvector

int k=EigenValue(OP,B,sym=true,sigma=sigma,value=ev,vector=eV,
                 tol=1e-10,maxit=0,ncv=0);

//      tol= the tolerance
//      maxit= the maximum iteration see arpack doc.
//      ncv see arpack doc. http://www.caam.rice.edu/software/ARPACK/
//      the return value is number of converged eigen value.

for (int i=0;i<k;i++)
{
    u1=eV[i];
    real gg = int2d(Th) (dx(u1)*dx(u1) + dy(u1)*dy(u1));
    real mm= int2d(Th) (u1*u1) ;
    cout << " ---- " << i << " " << ev[i] << " err= "
          << int2d(Th) (dx(u1)*dx(u1) + dy(u1)*dy(u1) - (ev[i])*u1*u1) << " --- " << endl;
    plot(eV[i],cmm="Eigen Vector "+i+" valeur =" + ev[i] ,wait=1,value=1);
}

```

The output of this example is:

```

Nb of edges on Mortars = 0
Nb of edges on Boundary = 80, neb = 80
Nb Of Nodes = 1681
Nb of DF = 1681
Real symmetric eigenvalue problem:  $A*x - B*x*\lambda$ 

```

```

Thanks to ARPACK++ class ARrcSymGenEig
Real symmetric eigenvalue problem:  $A*x - B*x*\lambda$ 
Shift and invert mode sigma=20

```

```

Dimension of the system          : 1681
Number of 'requested' eigenvalues : 20
Number of 'converged' eigenvalues : 20
Number of Arnoldi vectors generated: 41
Number of iterations taken       : 2

```

```

Eigenvalues:
lambda[1]: 5.0002
lambda[2]: 8.00074
lambda[3]: 10.0011
lambda[4]: 10.0011
lambda[5]: 13.002

```

```

lambda[6]: 13.0039
lambda[7]: 17.0046
lambda[8]: 17.0048
lambda[9]: 18.0083
lambda[10]: 20.0096
lambda[11]: 20.0096
lambda[12]: 25.014
lambda[13]: 25.0283
lambda[14]: 26.0159
lambda[15]: 26.0159
lambda[16]: 29.0258
lambda[17]: 29.0273
lambda[18]: 32.0449
lambda[19]: 34.049
lambda[20]: 34.0492

---- 0 5.0002 err= -0.000225891 ---
---- 1 8.00074 err= -0.000787446 ---
---- 2 10.0011 err= -0.00134596 ---
---- 3 10.0011 err= -0.00134619 ---
---- 4 13.002 err= -0.00227747 ---
---- 5 13.0039 err= -0.004179 ---
---- 6 17.0046 err= -0.00623649 ---
---- 7 17.0048 err= -0.00639952 ---
---- 8 18.0083 err= -0.00862954 ---
---- 9 20.0096 err= -0.0110483 ---
---- 10 20.0096 err= -0.0110696 ---
---- 11 25.014 err= -0.0154412 ---
---- 12 25.0283 err= -0.0291014 ---
---- 13 26.0159 err= -0.0218532 ---
---- 14 26.0159 err= -0.0218544 ---
---- 15 29.0258 err= -0.0311961 ---
---- 16 29.0273 err= -0.0326472 ---
---- 17 32.0449 err= -0.0457328 ---
---- 18 34.049 err= -0.0530978 ---
---- 19 34.0492 err= -0.0536275 ---

```

9.5 Evolution Problems

FreeFem++ also solves evolution problems such as the heat equation:

$$\begin{aligned}
 \frac{\partial u}{\partial t} - \mu \Delta u &= f \quad \text{in } \Omega \times]0, T[, \\
 u(\mathbf{x}, 0) &= u_0(\mathbf{x}) \quad \text{in } \Omega; \quad (\partial u / \partial n)(\mathbf{x}, t) = 0 \quad \text{on } \partial \Omega \times]0, T[.
 \end{aligned}
 \tag{9.26}$$

with a positive viscosity coefficient μ and homogeneous Neumann boundary conditions. We solve (9.26) by FEM in space and finite differences in time. We use the definition of the partial derivative of the solution in the time derivative,

$$\frac{\partial u}{\partial t}(x, y, t) = \lim_{\tau \rightarrow 0} \frac{u(x, y, t) - u(x, y, t - \tau)}{\tau}$$

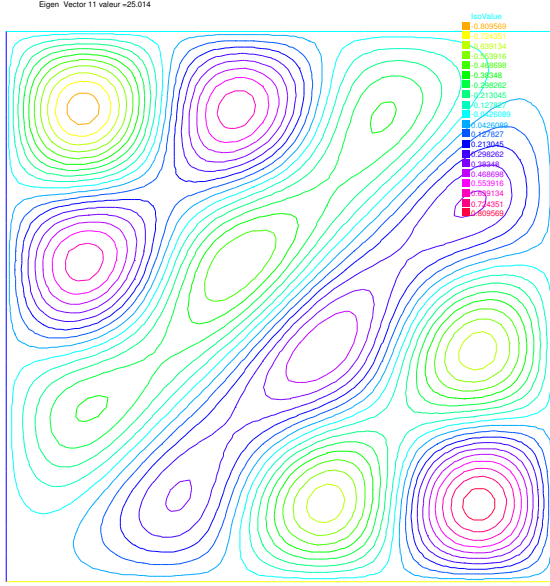


Figure 9.17: Isovalue of 11th eigenvector $u_{4,3} - u_{3,4}$

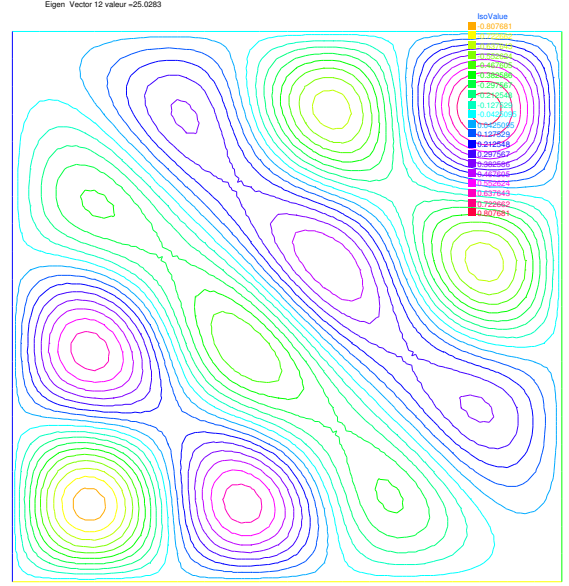


Figure 9.18: Isovalue of 12th eigenvector $u_{4,3} + u_{3,4}$

which indicates that $u^m(x, y) = u(x, y, m\tau)$ will satisfy approximately

$$\frac{\partial u}{\partial t}(x, y, m\tau) \simeq \frac{u^m(x, y) - u^{m-1}(x, y)}{\tau}$$

The time discretization of heat equation (9.27) is as follows:

$$\begin{aligned} \frac{u^{m+1} - u^m}{\tau} - \mu \Delta u^{m+1} &= f^{m+1} \quad \text{in } \Omega \\ u^0(\mathbf{x}) &= u_0(\mathbf{x}) \quad \text{in } \Omega; \quad \partial u^{m+1} / \partial n(\mathbf{x}) = 0 \quad \text{on } \partial\Omega, \quad \text{for all } m = 0, \dots, [T/\tau], \end{aligned} \quad (9.27)$$

which is so-called *backward Euler method* for (9.27). To obtain the variational formulation, multiply with the test function v both sides of the equation:

$$\int_{\Omega} \{u^{m+1}v - \tau \Delta u^{m+1}v\} = \int_{\Omega} \{u^m + \tau f^{m+1}\}v.$$

By the divergence theorem, we have

$$\int_{\Omega} \{u^{m+1}v + \tau \nabla u^{m+1} \cdot \nabla v\} - \int_{\partial\Omega} \tau (\partial u^{m+1} / \partial n) v = \int_{\Omega} \{u^m v + \tau f^{m+1}v\}.$$

By the boundary condition $\partial u^{m+1} / \partial n = 0$, it follows that

$$\int_{\Omega} \{u^{m+1}v + \tau \nabla u^{m+1} \cdot \nabla v\} - \int_{\Omega} \{u^m v + \tau f^{m+1}v\} = 0. \quad (9.28)$$

Using the identity just above, we can calculate the finite element approximation u_h^m of u^m in a step-by-step manner with respect to t .

Example 9.18 We now solve the following example with the exact solution $u(x, y, t) = tx^4$.

$$\begin{aligned} \frac{\partial u}{\partial t} - \mu \Delta u &= x^4 - \mu 12tx^2 \text{ in } \Omega \times]0, 3[, \Omega =]0, 1[^2 \\ u(x, y, 0) &= 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = t * x^4 \end{aligned}$$

```

//      heat equation  $\partial_t u = -\mu \Delta u = x^4 - \mu 12tx^2$ 

mesh Th=square(16,16);
fespace Vh(Th,P1);

Vh u,v,uu,f,g;
real dt = 0.1, mu = 0.01;
problem dHeat(u,v) =
  int2d(Th) ( u*v + dt*mu*(dx(u)*dx(v) + dy(u)*dy(v))
+ int2d(Th) (- uu*v - dt*f*v )
+ on(1,2,3,4,u=g);

real t = 0;
uu = 0;
for (int m=0;m<=3/dt;m++)
{
  t=t+dt;
  f = x^4-mu*t*12*x^2;
  g = t*x^4;
  dHeat;
  plot(u,wait=true);
  uu = u;
  cout <<"t="<<t<<"L^2-Error="<<sqrt( int2d(Th) ((u-t*x^4)^2) ) << endl;
}

```

In the last statement, the L^2 -error $\left(\int_{\Omega} |u - tx^4|^2\right)^{1/2}$ is calculated at $t = m\tau, \tau = 0.1$. At $t = 0.1$, the error is 0.000213269. The errors increase with m and 0.00628589 at $t = 3$. The iteration of the backward Euler (9.28) is made by **for loop** (see Section 4.11).

Note 9.2 The stiffness matrix in the loop is used over and over again. *FreeFem++* support reuses of stiffness matrix.

9.5.1 Mathematical Theory on Time Difference Approximations.

In this section, we show the advantage of implicit schemes. Let V, H be separable Hilbert space and V is dense in H . Let a be a continuous bilinear form over $V \times V$ with coercivity and symmetry. Then $\sqrt{a(v, v)}$ become equivalent to the norm $\|v\|$ of V .

Problem Ev(f, Ω): For a given $f \in L^2(0, T; V')$, $u^0 \in H$

$$\begin{aligned} \frac{d}{dt}(u(t), v) + a(u(t), v) &= (f(t), v) \quad \forall v \in V, \quad a.e. t \in [0, T] \\ u(0) &= u^0 \end{aligned} \quad (9.29)$$

where V' is the dual space of V . Then, there is an unique solution $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$.

Let us denote the time step by $\tau > 0$, $N_T = [T/\tau]$. For the discretization, we put $u^n = u(n\tau)$ and consider the time difference for each $\theta \in [0, 1]$

$$\begin{aligned} \frac{1}{\tau} (u_h^{n+1} - u_h^n, \phi_i) + a(u_h^{n+\theta}, \phi_i) &= \langle f^{n+\theta}, \phi_i \rangle \\ i &= 1, \dots, m, \quad n = 0, \dots, N_T \\ u_h^{n+\theta} &= \theta u_h^{n+1} + (1 - \theta) u_h^n, \quad f^{n+\theta} = \theta f^{n+1} + (1 - \theta) f^n \end{aligned} \quad (9.30)$$

Formula (9.30) is the *forward Euler scheme* if $\theta = 0$, *Crank-Nicolson scheme* if $\theta = 1/2$, the *backward Euler scheme* if $\theta = 1$.

Unknown vectors $u^n = (u_h^1, \dots, u_h^M)^T$ in

$$u_h^n(x) = u_1^n \phi_1(x) + \dots + u_m^n \phi_m(x), \quad u_1^n, \dots, u_m^n \in \mathbb{R}$$

are obtained from solving the matrix

$$\begin{aligned} (M + \theta \tau A) u^{n+1} &= \{M - (1 - \theta) \tau A\} u^n + \tau \{\theta f^{n+1} + (1 - \theta) f^n\} \\ M &= (m_{ij}), \quad m_{ij} = (\phi_j, \phi_i), \quad A = (a_{ij}), \quad a_{ij} = a(\phi_j, \phi_i) \end{aligned} \quad (9.31)$$

Refer [22, pp.70–75] for solvability of (9.31). The stability of (9.31) is in [22, Theorem 2.13]:

Let $\{\mathcal{T}_h\}_{h \downarrow 0}$ be regular triangulations (see Section 5.4). Then there is a number $c_0 > 0$ independent of h such that,

$$|u_h^n|^2 \leq \begin{cases} \frac{1}{\delta} \left\{ |u_h^0|^2 + \tau \sum_{k=0}^{n-1} \|f^{k+\theta}\|_{V_h'}^2 \right\} & \theta \in [0, 1/2) \\ |u_h^0|^2 + \tau \sum_{k=0}^{n-1} \|f^{k+\theta}\|_{V_h'}^2 & \theta \in [1/2, 1] \end{cases} \quad (9.32)$$

if the following are satisfied:

1. When $\theta \in [0, 1/2)$, then we can take a time step τ in such a way that

$$\tau < \frac{2(1 - \delta)}{(1 - 2\theta)c_0^2} h^2 \quad (9.33)$$

for arbitrary $\delta \in (0, 1)$.

2. When $1/2 \leq \theta \leq 1$, we can take τ arbitrary.

Example 9.19

```

mesh Th=square(12,12);
fespace Vh(Th,P1);
fespace Ph(Th,P0);

Ph h = hTriangle; // mesh sizes for each triangle
real tau = 0.1, theta=0.;
func real f(real t) {
    return x^2*(x-1)^2 + t*(-2 + 12*x - 11*x^2 - 2*x^3 + x^4);
}
ofstream out("err02.csv"); // file to store calculations
out << "mesh size = "<<h[0].max<<", time step = "<<tau<<endl;
for (int n=0;n<5/tau;n++) \
    out<<n*tau<<",";
out << endl;

```

```

Vh u,v,oldU;
Vh f1, f0;
problem aTau(u,v) =
  int2d(Th) ( u*v + theta*tau*(dx(u)*dx(v) + dy(u)*dy(v) + u*v))
- int2d(Th) (oldU*v - (1-theta)*tau*(dx(oldU)*dx(v)+dy(oldU)*dy(v)+oldU*v))
- int2d(Th) (tau*( theta*f1+(1-theta)*f0 )*v );

while (theta <= 1.0) {
  real t = 0, T=3; // from t=0 to T
  oldU = 0; // u(x,y,0)=0
  out << theta << ", ";
  for (int n=0;n<T/tau;n++) {
    t = t+tau;
    f0 = f(n*tau); f1 = f((n+1)*tau);
    aTau;
    oldU = u;
    plot (u);
    Vh uex = t*x^2*(1-x)^2; // exact sol.= tx^2(1-x)^2
    Vh err = u - uex; // err=FE-sol - exact
    out<< abs(err[]).max()/abs(uex[]).max) << ", "; // ||err||_{L^\infty(\Omega)}/||u_{ex}||_{L^\infty(\Omega)}
  }
  out << endl;
  theta = theta + 0.1;
}

```

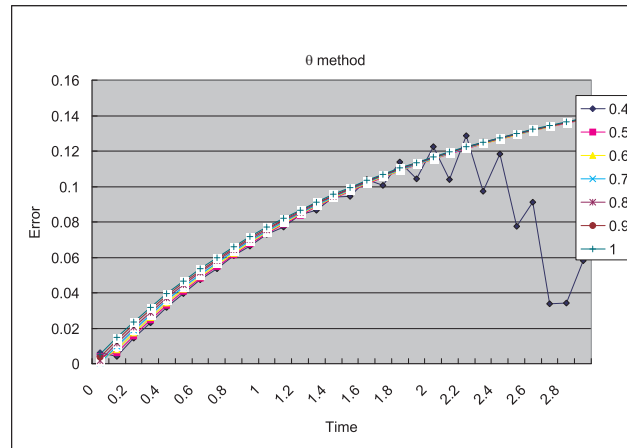


Figure 9.19: $\max_{x \in \Omega} |u_h^n(\theta) - u_{ex}(n\tau)| / \max_{x \in \Omega} |u_{ex}(n\tau)|$ at $n = 0, 1, \dots, 29$

We can see in Fig. 9.19 that $u_h^n(\theta)$ become unstable at $\theta = 0.4$, and figures are omitted in the case $\theta < 0.4$.

9.5.2 Convection

The hyperbolic equation

$$\partial_t u + \alpha \cdot \nabla u = f; \quad \text{for a vector-valued function } \alpha, \quad (9.34)$$

appears frequently in scientific problems, for example in the Navier-Stokes equations, in the Convection-Diffusion equation, etc.

In the case of 1-dimensional space, we can easily find the general solution $(x, t) \mapsto u(x, t) = u^0(x - \alpha t)$ of the following equation, if α is constant,

$$\partial_t u + \alpha \partial_x u = 0, \quad u(x, 0) = u^0(x), \quad (9.35)$$

because $\partial_t u + \alpha \partial_x u = -\alpha \dot{u}^0 + \alpha \dot{u}^0 = 0$, where $\dot{u}^0 = du^0(x)/dx$. Even if α is not constant, the construction works on similar principles. One begins with the ordinary differential equation (with the convention that α is prolonged by zero apart from $(0, L) \times (0, T)$):

$$\dot{X}(\tau) = +\alpha(X(\tau), \tau), \quad \tau \in (0, t) \quad X(t) = x$$

In this equation τ is the variable and x, t are parameters, and we denote the solution by $X_{x,t}(\tau)$. Then it is noticed that $(x, t) \rightarrow v(X(\tau), \tau)$ in $\tau = t$ satisfies the equation

$$\partial_t v + \alpha \partial_x v = \partial_t X \dot{v} + \alpha \partial_x X \dot{v} = 0$$

and by the definition $\partial_t X = \dot{X} = +\alpha$ and $\partial_x X = \partial_x x$ in $\tau = t$, because if $\tau = t$ we have $X(\tau) = x$. The general solution of (9.35) is thus the value of the boundary condition in $X_{x,t}(0)$, that is to say $u(x, t) = u^0(X_{x,t}(0))$ where $X_{x,t}(0)$ is on the x axis, $u(x, t) = u^0(X_{x,t}(0))$ if $X_{x,t}(0)$ is on the axis of t .

In higher dimension $\Omega \subset R^d$, $d = 2, 3$, the equation for the convection is written

$$\partial_t u + \boldsymbol{\alpha} \cdot \nabla u = 0 \text{ in } \Omega \times (0, T)$$

where $\boldsymbol{\alpha}(x, t) \in R^d$. FreeFem++ implements the Characteristic-Galerkin method for convection operators. Recall that the equation (9.34) can be discretized as

$$\frac{Du}{Dt} = f \text{ i.e. } \frac{du}{dt}(X(t), t) = f(X(t), t) \text{ where } \frac{dX}{dt}(t) = \boldsymbol{\alpha}(X(t), t)$$

where D is the total derivative operator. So a good scheme is one step of backward convection by the method of Characteristics-Galerkin

$$\frac{1}{\tau} (u^{m+1}(x) - u^m(X^m(x))) = f^m(x) \quad (9.36)$$

where $X^m(x)$ is an approximation of the solution at $t = m\tau$ of the ordinary differential equation

$$\frac{d\mathbf{X}}{dt}(t) = \boldsymbol{\alpha}^m(\mathbf{X}(t)), \quad \mathbf{X}((m+1)\tau) = x.$$

where $\boldsymbol{\alpha}^m(x) = (\alpha_1(x, m\tau), \alpha_2(x, m\tau))$. Because, by Taylor's expansion, we have

$$\begin{aligned} u^m(\mathbf{X}(m\tau)) &= u^m(\mathbf{X}((m+1)\tau)) - \tau \sum_{i=1}^d \frac{\partial u^m}{\partial x_i}(\mathbf{X}((m+1)\tau)) \frac{\partial X_i}{\partial t}((m+1)\tau) + o(\tau) \\ &= u^m(x) - \tau \boldsymbol{\alpha}^m(x) \cdot \nabla u^m(x) + o(\tau) \end{aligned} \quad (9.37)$$

where $X_i(t)$ are the i -th component of $\mathbf{X}(t)$, $u^m(x) = u(x, m\tau)$ and we used the chain rule and $x = \mathbf{X}((m+1)\tau)$. From (9.37), it follows that

$$u^m(X^m(x)) = u^m(x) - \tau \boldsymbol{\alpha}^m(x) \cdot \nabla u^m(x) + o(\tau). \quad (9.38)$$

Also we apply Taylor's expansion for $t \mapsto u^m(x - \boldsymbol{\alpha}^m(x)t)$, $0 \leq t \leq \tau$, then

$$u^m(x - \boldsymbol{\alpha}\tau) = u^m(x) - \tau \boldsymbol{\alpha}^m(x) \cdot \nabla u^m(x) + o(\tau).$$

Putting

$$\text{convect}(\alpha, -\tau, u^m) \approx u^m(x - \alpha^m \tau),$$

we can get the approximation

$$u^m(X^m(x)) \approx \text{convect}([a_1^m, a_2^m], -\tau, u^m) \text{ by } X^m \approx x \mapsto x - \tau[a_1^m(x), a_2^m(x)].$$

A classical convection problem is that of the “rotating bell” (quoted from [14][p.16]). Let Ω be the unit disk centered at 0, with its center rotating with speed $\alpha_1 = y$, $\alpha_2 = -x$. We consider the problem (9.34) with $f = 0$ and the initial condition $u(x, 0) = u^0(x)$, that is, from (9.36)

$$u^{m+1}(x) = u^m(X^m(x)) \approx \text{convect}(\alpha, -\tau, u^m).$$

The exact solution is $u(x, t) = u(\mathbf{X}(t))$ where \mathbf{X} equals x rotated around the origin by an angle $\theta = -t$ (rotate in clockwise). So, if u^0 in a 3D perspective looks like a bell, then u will have exactly the same shape, but rotated by the same amount. The program consists in solving the equation until $T = 2\pi$, that is for a full revolution and to compare the final solution with the initial one; they should be equal.

```
Example 9.20 (convect.edp) border C(t=0, 2*pi) { x=cos(t); y=sin(t); }; // the
unit circle
mesh Th = buildmesh(C(70)); // triangulates the disk
fespace Vh(Th, P1);
Vh u0 = exp(-10*((x-0.3)^2 + (y-0.3)^2)); // give u^0

real dt = 0.17, t=0; // time step
Vh a1 = -y, a2 = x; // rotation velocity
Vh u; // u^{m+1}
for (int m=0; m<2*pi/dt ; m++) {
  t += dt;
  u=convect([a1, a2], -dt, u0); // u^{m+1} = u^m(X^m(x))
  u0=u; // m++
  plot(u, cmm=" t="+t + ", min=" + u[].min + ", max=" + u[].max, wait=0);
};
```

Note 9.3 The scheme *convect* is unconditionally stable, then the bell become lower and lower (the maximum of u^{37} is 0.406 as shown in Fig. 9.21).

9.5.3 2D Black-Scholes equation for an European Put option

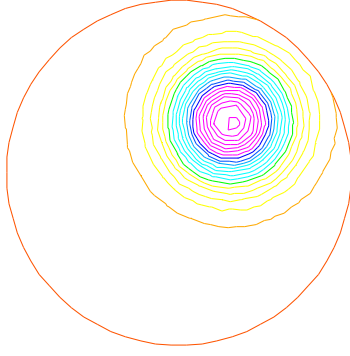
In mathematical finance, an option on two assets is modeled by a Black-Scholes equations in two space variables, (see for example Wilmott et al[39] or Achdou et al [3]).

$$\begin{aligned} \partial_t u + \frac{(\sigma_1 x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\sigma_2 y)^2}{2} \frac{\partial^2 u}{\partial y^2} \\ + \rho xy \frac{\partial^2 u}{\partial x \partial y} + rS_1 \frac{\partial u}{\partial x} + rS_2 \frac{\partial u}{\partial y} - rP = 0 \end{aligned} \quad (9.39)$$

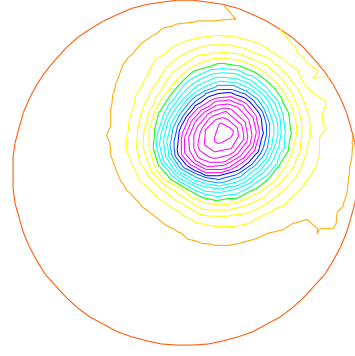
which is to be integrated in $(0, T) \times \mathbb{R}^+ \times \mathbb{R}^+$ subject to, in the case of a put

$$u(x, y, T) = (K - \max(x, y))^+. \quad (9.40)$$

convection: t=0, min=1.55289e-09, max=0.983612

Figure 9.20: $u^0 = e^{-10((x-0.3)^2 + (y-0.3)^2)}$

convection: t=6.29, min=1.55289e-09, max=0.40659m=37

Figure 9.21: The bell at $t = 6.29$

Boundary conditions for this problem may not be so easy to device. As in the one dimensional case the PDE contains boundary conditions on the axis $x_1 = 0$ and on the axis $x_2 = 0$, namely two one dimensional Black-Scholes equations driven respectively by the data $u(0, +\infty, T)$ and $u(+\infty, 0, T)$. These will be automatically accounted for because they are embedded in the PDE. So if we do nothing in the variational form (i.e. if we take a Neumann boundary condition at these two axis in the strong form) there will be no disturbance to these. At infinity in one of the variable, as in 1D, it makes sense to impose $u = 0$. We take

$$\sigma_1 = 0.3, \quad \sigma_2 = 0.3, \quad \rho = 0.3, \quad r = 0.05, \quad K = 40, \quad T = 0.5 \quad (9.41)$$

An implicit Euler scheme is used and a mesh adaptation is done every 10 time steps. To have an unconditionally stable scheme, the first order terms are treated by the Characteristic Galerkin method, which, roughly, approximates

$$\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} \approx \frac{1}{\tau} (u^{n+1}(x) - u^n(x - \alpha \tau)) \quad (9.42)$$

Example 9.21 *[BlackSchol.edp]*

```

//      file BlackScholes2D.edp

int m=30,L=80,LL=80, j=100;
real sigx=0.3, sigy=0.3, rho=0.3, r=0.05, K=40, dt=0.01;
mesh th=square(m,m, [L*x,LL*y]);
fespace Vh(th,P1);

Vh u=max(K-max(x,y),0.);
Vh xveloc, yveloc, v,uold;

for (int n=0; n*dt <= 1.0; n++)
{
  if(j>20) { th = adaptmesh(th,u,verbosity=1,absserror=1,nbjacoby=2,
    err=0.001, nbvx=5000, omega=1.8, ratio=1.8, nbsmooth=3,
    splitpbedge=1, maxsubdiv=5,rescaling=1) ;
    j=0;
    xveloc = -x*r+x*sigx^2+x*rho*sigx*sigy/2;
    yveloc = -y*r+y*sigy^2+y*rho*sigx*sigy/2;
  }
}
```

```

    u=u;
  };
  uold=u;
  solve eq1(u,v,init=j,solver=LU) = int2d(th) ( u*v*(r+1/dt)
    + dx(u)*dx(v)*(x*sigx)^2/2 + dy(u)*dy(v)*(y*sigy)^2/2
    + (dy(u)*dx(v) + dx(u)*dy(v))*rho*sigx*sigy*x*y/2)
    - int2d(th) ( v*convect([xveloc,yveloc],dt,w)/dt) + on(2,3,u=0);

  j=j+1;
};
plot(u,wait=1,value=1);

```

Results are shown on Fig. 9.21).

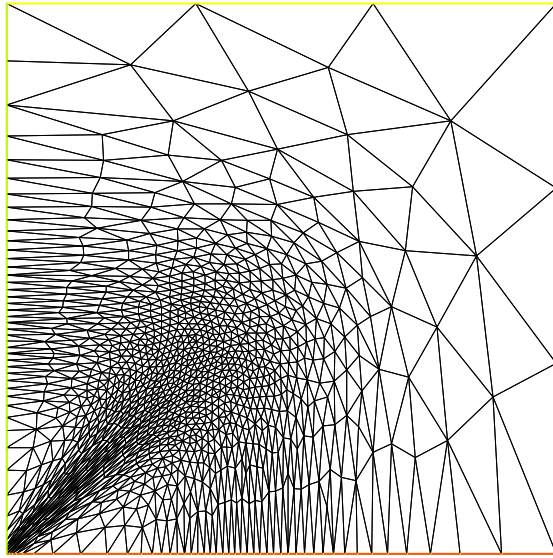


Figure 9.22: The adapted triangulation

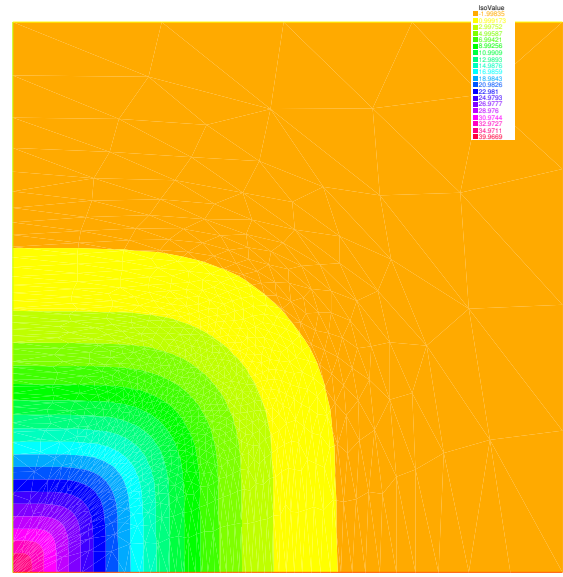


Figure 9.23: The level line of the European basquet put option

9.6 Navier-Stokes Equation

9.6.1 Stokes and Navier-Stokes

The Stokes equations are: for a given $\mathbf{f} \in L^2(\Omega)^2$,

$$\left. \begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\} \quad \text{in } \Omega \quad (9.43)$$

where $\mathbf{u} = (u_1, u_2)$ is the velocity vector and p the pressure. For simplicity, let us choose Dirichlet boundary conditions on the velocity, $\mathbf{u} = \mathbf{u}_\Gamma$ on Γ .

In Temam [Theorem 2.2], there is a weak form of (9.43): Find $\mathbf{v} = (v_1, v_2) \in \mathbf{V}(\Omega)$

$$\mathbf{V}(\Omega) = \{\mathbf{w} \in H_0^1(\Omega)^2 \mid \operatorname{div} \mathbf{w} = 0\}$$

which satisfy

$$\sum_{i=1}^2 \int_{\Omega} \nabla u_i \cdot \nabla v_i = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \quad \text{for all } \mathbf{v} \in \mathbf{V}$$

Here it is used the existence $p \in H^1(\Omega)$ such that $\mathbf{u} = \nabla p$, if

$$\int_{\Omega} \mathbf{u} \cdot \mathbf{v} = 0 \quad \text{for all } \mathbf{v} \in \mathbf{V}$$

Another weak form is derived as follows: We put

$$\mathbf{V} = H_0^1(\Omega)^2; \quad W = \left\{ q \in L^2(\Omega) \mid \int_{\Omega} q = 0 \right\}$$

By multiplying the first equation in (9.43) with $\mathbf{v} \in \mathbf{V}$ and the second with $q \in W$, subsequent integration over Ω , and an application of Green's formula, we have

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \int_{\Omega} \operatorname{div} \mathbf{v} p &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \\ \int_{\Omega} \operatorname{div} \mathbf{u} q &= 0 \end{aligned}$$

This yields the weak form of (9.43): Find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad (9.44)$$

$$b(\mathbf{u}, q) = 0 \quad (9.45)$$

for all $(\mathbf{v}, q) \in \mathbf{V} \times W$, where

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} = \sum_{i=1}^2 \int_{\Omega} \nabla u_i \cdot \nabla v_i \quad (9.46)$$

$$b(\mathbf{u}, q) = - \int_{\Omega} \operatorname{div} \mathbf{u} q \quad (9.47)$$

Now, we consider finite element spaces $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$, and we assume the following basis functions

$$\begin{aligned} \mathbf{V}_h &= V_h \times V_h, \quad V_h = \{v_h \mid v_h = v_1 \phi_1 + \cdots + v_{M_V} \phi_{M_V}\}, \\ W_h &= \{q_h \mid q_h = q_1 \varphi_1 + \cdots + q_{M_W} \varphi_{M_W}\} \end{aligned}$$

The discrete weak form is: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in W_h \end{aligned} \quad (9.48)$$

Note 9.4 Assume that:

1. There is a constant $\alpha_h > 0$ such that

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \alpha \|\mathbf{v}_h\|_{1,\Omega}^2 \quad \text{for all } \mathbf{v}_h \in Z_h$$

where

$$Z_h = \{\mathbf{v}_h \in \mathbf{V}_h \mid b(\mathbf{w}_h, q_h) = 0 \quad \text{for all } q_h \in W_h\}$$

2. There is a constant $\beta_h > 0$ such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \beta_h \|q_h\|_{0,\Omega} \quad \text{for all } q_h \in W_h$$

Then we have an unique solution (\mathbf{u}_h, p_h) of (9.48) satisfying

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \inf_{q_h \in W_h} \|p - q_h\|_{0,\Omega} \right)$$

with a constant $C > 0$ (see e.g. [20, Theorem 10.4]).

Let us denote that

$$\begin{aligned} \mathbf{A} &= (A_{ij}), \quad A_{ij} = \int_{\Omega} \nabla \phi_j \cdot \nabla \phi_i \quad i, j = 1, \dots, M_{\mathbf{V}} \\ \mathbf{B} &= (Bx_{ij}, By_{ij}), \quad Bx_{ij} = - \int_{\Omega} \partial \phi_j / \partial x \varphi_i \quad By_{ij} = - \int_{\Omega} \partial \phi_j / \partial y \varphi_i \\ &\quad i = 1, \dots, M_W; j = 1, \dots, M_V \end{aligned} \quad (9.49)$$

then (9.48) is written by

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_h \\ \{p_h\} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_h \\ 0 \end{pmatrix} \quad (9.50)$$

where

$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \mathbf{B}^* = \begin{Bmatrix} Bx^T \\ By^T \end{Bmatrix} \quad \mathbf{U}_h = \begin{Bmatrix} \{u_{1,h}\} \\ \{u_{2,h}\} \end{Bmatrix} \quad \mathbf{F}_h = \begin{Bmatrix} \{\int_{\Omega} f_1 \phi_i\} \\ \{\int_{\Omega} f_2 \phi_i\} \end{Bmatrix}$$

Penalty method: This method consists of replacing (9.48) by a more regular problem: Find $(\mathbf{v}_h^\epsilon, p_h^\epsilon) \in \mathbf{V}_h \times \tilde{W}_h$ satisfying

$$\begin{aligned} a(\mathbf{u}_h^\epsilon, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^\epsilon) &= (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h^\epsilon, q_h) - \epsilon(p_h^\epsilon, q_h) &= 0, \quad \forall q_h \in \tilde{W}_h \end{aligned} \quad (9.51)$$

where $\tilde{W}_h \subset L^2(\Omega)$. Formally, we have

$$\operatorname{div} \mathbf{u}_h^\epsilon = \epsilon p_h^\epsilon$$

and the corresponding algebraic problem

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & -\epsilon I \end{pmatrix} \begin{pmatrix} \mathbf{U}_h^\epsilon \\ \{p_h^\epsilon\} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_h \\ 0 \end{pmatrix}$$

Note 9.5 We can eliminate $p_h^\epsilon = (1/\epsilon)BU_h^\epsilon$ to obtain

$$(A + (1/\epsilon)B^*B)U_h^\epsilon = F_h^\epsilon \quad (9.52)$$

Since the matrix $A + (1/\epsilon)B^*B$ is symmetric, positive-definite, and sparse, (9.52) can be solved by known technique. There is a constant $C > 0$ independent of ϵ such that

$$\|u_h - u_h^\epsilon\|_{1,\Omega} + \|p_h - p_h^\epsilon\|_{0,\Omega} \leq C\epsilon$$

(see e.g. [20, 17.2])

Example 9.22 (Cavity.edp) The driven cavity flow problem is solved first at zero Reynolds number (Stokes flow) and then at Reynolds 100. The velocity pressure formulation is used first and then the calculation is repeated with the stream function vorticity formulation.

We solve the driven cavity problem by the penalty method (9.51) where $u_\Gamma \cdot n = 0$ and $u_\Gamma \cdot s = 1$ on the top boundary and zero elsewhere (n is the unit normal to Γ , and s the unit tangent to Γ). The mesh is constructed by

```
mesh Th=square(8,8);
```

We use a classical Taylor-Hood element technic to solve the problem:

The velocity is approximated with the P_2 FE (X_h space), and the the pressure is approximated with the P_1 FE (M_h space),

where

$$X_h = \{v \in H^1([0,1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2\}$$

and

$$M_h = \{v \in H^1([0,1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1\}$$

The FE spaces and functions are constructed by

```
fespace Xh(Th,P2);           // definition of the velocity component space
fespace Mh(Th,P1);           // definition of the pressure space
Xh u2,v2;
Xh u1,v1;
Mh p,q;
```

The Stokes operator is implemented as a system-solve for the velocity ($u1, u2$) and the pressure p . The test function for the velocity is ($v1, v2$) and q for the pressure, so the variational form (9.48) in freefem language is:

```
solve Stokes (u1,u2,p,v1,v2,q,solver=Crout) =
  int2d(Th) ( ( dx(u1)*dx(v1) + dy(u1)*dy(v1)
    + dx(u2)*dx(v2) + dy(u2)*dy(v2) )
    - p*q*(0.000001)
    - p*dx(v1) - p*dy(v2)
    - dx(u1)*q - dy(u2)*q
  )
+ on(3,u1=1,u2=0)
+ on(1,2,4,u1=0,u2=0);           // see Section 5.1.1 for labels 1,2,3,4
```

Each unknown has its own boundary conditions.

If the streamlines are required, they can be computed by finding ψ such that $\text{rot}\psi = u$ or better,

$$-\Delta\psi = \nabla \times u$$

```
Xh psi,phi;

solve streamlines(psi,phi) =
  int2d(Th) ( dx(psi)*dx(phi) + dy(psi)*dy(phi))
+  int2d(Th) ( -phi*(dy(u1)-dx(u2)))
+  on(1,2,3,4,psi=0);
```

Now the Navier-Stokes equations are solved

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0$$

with the same boundary conditions and with initial conditions $u = 0$.

This is implemented by using the convection operator `convect` for the term $\frac{\partial u}{\partial t} + u \cdot \nabla u$, giving a discretization in time

$$\begin{aligned} \frac{1}{\tau}(u^{n+1} - u^n \circ X^n) - \nu \Delta u^{n+1} + \nabla p^{n+1} &= 0, \\ \nabla \cdot u^{n+1} &= 0 \end{aligned} \quad (9.53)$$

The term $u^n \circ X^n(x) \approx u^n(x - u^n(x)\tau)$ will be computed by the operator “`convect`”, so we obtain

```
int i=0;
real nu=1./100.;
real dt=0.1;
real alpha=1/dt;

Xh up1,up2;

problem NS (u1,u2,p,v1,v2,q,solver=Croux,init=i) =
  int2d(Th) (
    alpha*( u1*v1 + u2*v2)
    + nu * ( dx(u1)*dx(v1) + dy(u1)*dy(v1)
    + dx(u2)*dx(v2) + dy(u2)*dy(v2) )
    - p*q*(0.000001)
    - p*dx(v1) - p*dy(v2)
    - dx(u1)*q - dy(u2)*q
  )
+  int2d(Th) ( -alpha*
    convect([up1,up2],-dt,up1)*v1 -alpha*convect([up1,up2],-dt,up2)*v2 )
+  on(3,u1=1,u2=0)
+  on(1,2,4,u1=0,u2=0)
;

for (i=0;i<=10;i++)
{
  up1=u1;
```

```

up2=u2;
NS;
if ( !(i % 10)) // plot every 10 iteration
    plot(coef=0.2,cmm=" [u1,u2] and p ",p,[u1,u2]);
} ;

```

Notice that the stiffness matrices are reused (keyword `init=i`)

9.6.2 Uzawa Algorithm and Conjugate Gradients

We solve Stokes problem without penalty. The classical iterative method of Uzawa is described by the algorithm (see e.g.[20, 17.3], [29, 13] or [30, 13]):

Initialize: Let p_h^0 be an arbitrary chosen element of $L^2(\Omega)$.

Calculate \mathbf{u}_h : Once p_h^n is known, \mathbf{v}_h^n is the solution of

$$\mathbf{u}_h^n = A^{-1}(\mathbf{f}_h - \mathbf{B}^* p_h^n)$$

Advance p_h : Let p_h^{n+1} be defined by

$$p_h^{n+1} = p_h^n + \rho_n \mathbf{B} \mathbf{u}_h^n$$

There is a constant $\alpha > 0$ such that $\alpha \leq \rho_n \leq 2$ for each n , then \mathbf{u}_h^n converges to the solution \mathbf{u}_h , and then $\mathbf{B} \mathbf{v}_h^n \rightarrow 0$ as $n \rightarrow \infty$ from the *Advance p_h* . This method in general converges quite slowly. First we define mesh, and the Taylor-Hood approximation. So X_h is the velocity space, and M_h is the pressure space.

Example 9.23 (StokesUzawa.edp)

```

mesh Th=square(10,10);
fespace Xh(Th,P2),Mh(Th,P1);
Xh u1,u2,v1,v2;
Mh p,q,ppp; // ppp is a working pressure

varf bx(u1,q) = int2d(Th) ( -(dx(u1)*q) );
varf by(u1,q) = int2d(Th) ( -(dy(u1)*q) );
varf a(u1,u2)= int2d(Th) ( dx(u1)*dx(u2) + dy(u1)*dy(u2) )
                + on(3,u1=1) + on(1,2,4,u1=0) ;
                // remark: put the on(3,u1=1) before on(1,2,4,u1=0)
                // because we want zero on intersection %

matrix A= a(Xh,Xh,solver=CG);
matrix Bx= bx(Xh,Mh); // B = (Bx By)
matrix By= by(Xh,Mh);

Xh bc1; bc1[] = a(0,Xh); // boundary condition contribution on u1
Xh bc2; bc2[] = 0; // no boundary condition contribution on u2
Xh b;

```

$p_h^n \rightarrow \mathbf{B} A^{-1}(-\mathbf{B}^* p_h^n) = -\text{div} \mathbf{u}_h$ is realized as the function `divup`.

```

func real[int] divup(real[int] & pp)
{

```

```

//      compute u1(pp)
b[]  = Bx'*pp; b[] *=-1; b[] += bc1[] ;    u1[] = A^-1*b[];
//      compute u2(pp)
b[]  = By'*pp; b[] *=-1; b[] += bc2[] ;    u2[] = A^-1*b[];
//       $u^n = A^{-1}(Bx^T p^n \quad By^T p^n)^T$ 
ppp[] = Bx*u1[]; //      ppp = Bxu1
ppp[] += By*u2[]; //      +Byu2
return ppp[] ;
};

```

Call now the conjugate gradient algorithm:

```

p=0;q=0; //       $p_h^0 = 0$ 
LinearCG(divup,p[],eps=1.e-6,nbiter=50); //       $p_h^{n+1} = p_h^n + Bu_h^n$ 
//      if  $n > 50$  or  $|p_h^{n+1} - p_h^n| \leq 10^{-6}$ , then the loop end.
divup(p[]); //      compute the final solution

plot([u1,u2],p,wait=1,value=true,coef=0.1);

```

9.6.3 NSUzawaCahouetChabart.edp

In this example we solve the Navier-Stokes equation past a cylinder with the Uzawa algorithm preconditioned by the Cahouet-Chabart method (see [31] for all the details).

The idea of the preconditioner is that in a periodic domain, all differential operators commute and the Uzawa algorithm comes to solving the linear operator $\nabla \cdot ((\alpha Id + \nu \Delta)^{-1} \nabla$, where Id is the identity operator. So the preconditioner suggested is $\alpha \Delta^{-1} + \nu Id$.

To implement this, we do

Example 9.24 (NSUzawaCahouetChabart.edp)

```

real D=0.1, H=0.41;
real cx0 = 0.2, cy0 = 0.2; //      center of cyl.
real xa = 0.15, ya=0.2, xe = 0.25, ye =0.2;
border fr1(t=0,2.2){x=t; y=0; label=1;}
border fr2(t=0,H){x=2.2; y=t; label=2;}
border fr3(t=2.2,0){x=t; y=H; label=1;}
border fr4(t=H,0){x=0; y=t; label=1;}
border fr5(t=2*pi,0){x=cx0+D*sin(t)/2; y=cy0+D*cos(t)/2; label=3;}
int nn=15;

mesh Th=buildmesh(fr1(5*nn)+fr2(nn)+fr3(5*nn)+fr4(nn)+fr5(-nn*3));
real Um= 1.5; //      max velocity (Rey 100)
func Ub = Um*2./3.;
real nu = 1e-3;
real Rey = Ub*D/nu; //      Boundary condition

func U1 = 4.*Um*y*(H-y)/(H*H) ;
func U2 = 0. ;

real T=2,t=0;
real dt = D/nn/Um; //      CFL = 1
cout << " dt = " << dt << endl;
real alpha=1/dt, epspq=1e-10;

```



```

fespace Mh(Th,[P1]);
fespace Xh(Th,[P2]);
fespace Wh(Th,[P1dc]);
macro grad(u) [dx(u),dy(u)] //
macro div(u1,u2) (dx(u1)+dy(u2)) //

varf von1([u1,u2,p],[v1,v2,q]) = on(3,u1=0,u2=0) + on(1,u1=U1,u2=U2);

// remark : the value 100 in next line is manually fitted, because free
outlet.
varf vA(p,q) = int2d(Th)((grad( p ) ' * grad(q)) ) + int1d(Th,2)(100*p*q) ;

varf vM(p,q) = int2d(Th,qft=qf2pT)( p*q ) + on(2,p=0);

varf vu([u1],[v1]) = int2d(Th)(alpha*(u1*v1)+nu*(grad(u1)'*grad(v1) ))
                    + on(1,3,u1=0) ;
varf vu1([p],[v1]) = int2d(Th)(p*dx(v1)) ;
varf vu2([p],[v1]) = int2d(Th)(p*dy(v1)) ;

matrix pAM=vM(Mh,Mh,solver=UMFPACK);
matrix pAA=vA(Mh,Mh,solver=UMFPACK);
matrix AU=vu(Xh,Xh,solver=UMFPACK);
matrix B1=vu1(Mh,Xh);
matrix B2=vu2(Mh,Xh);
Xh u1,u2;
Mh p;
varf vonu1([u1],[v1]) = on(1,u1=U1) + on(3,u1=0);
varf vonu2([u1],[v1]) = on(1,u1=U2) + on(3,u1=0);

real[int] brhs1 = vonu1(0,Xh);
real[int] brhs2 = vonu2(0,Xh);

varf vrhs1(uu,vv) = int2d(Th) (convect ([u1,u2],-dt,u1)*vv*alpha)+vonu1 ;
varf vrhs2(v2,v1) = int2d(Th) (convect ([u1,u2],-dt,u2)*v1*alpha)+vonu2;

```

The functions to define Uzawa and the preconditioner part.

```

func real[int] JUzawa(real[int] & pp)
{
real[int] b1=brhs1; b1 += B1*pp;
real[int] b2=brhs2; b2 += B2*pp;
u1[] = AU^-1 * b1;
u2[] = AU^-1 * b2;
pp = B1'*u1[];
pp += B2'*u2[];
pp = -pp;
return pp;
}

func real[int] Precon(real[int] & p)

```

```

{
    real[int] pa= pAA^-1*p;
    real[int] pm= pAM^-1*p;
    real[int] pp= alpha*pa+nu*pm;
    return pp;
}

```

The loop in time. Warning with the stop test of the conjugate gradient, because we start from the previous solution and the end the previous solution is close to the final solution, don't take a relative stop test to the first residual, take an absolute stop test (negative here)

```

verbosity = 0;
p=0;

Wh w; // to store vorticity ..

real eps=1e-6;
int ndt = T/dt;
for(int i=0;i<ndt;++i)
{
    brhs1 = vrhs1(0,Xh);
    brhs2 = vrhs2(0,Xh);
    int res=LinearCG(JUzawa,p[],precon=Precon,nbiter=100,verbosity=10,veps=eps);
    assert(res==1) ;
    eps = -abs(eps);
    w = -dy(u1)+dx(u2);
    plot(w,fill=1,wait=0, nbiso=40);

    dt = min(dt,T-t);
    t += dt;
    if( dt < 1e-10*T) break;
}
plot(w,fill=1,wait=0, nbiso=40,ps="NScahouetChabart"); // see fig. 9.24

cout << " u1 max " << u1[].linfty
      << " u2 max " << u2[].linfty
      << " p max = " << p[].max << endl;

```

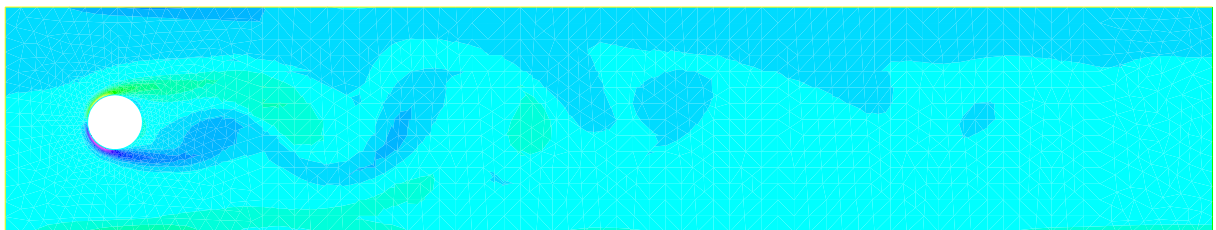


Figure 9.24: The vorticity at Reynolds number 100 a time 2s with the Cahouet-Chabart method.

9.7 Variational inequality

We present, a classical example of variational inequality.

Let us denote $\mathcal{C} = \{u \in H_0^1(\Omega), u \leq g\}$

The problem is :

$$u = \arg \min_{u \in \mathcal{C}} J(u) = \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u - \int_{\Omega} f u$$

where f and g are given function.

The solution is a projection on the convex \mathcal{C} of f^* for the scalar product $((v, w)) = \int_{\Omega} \nabla v \cdot \nabla w$ of $H_0^1(\Omega)$ where f^* is solution of $((f^*, v)) = \int_{\Omega} f v, \forall v \in H_0^1(\Omega)$. The projection on a convex satisfy clearly $\forall v \in \mathcal{C}, ((u - v, u - \tilde{f})) \leq 0$, and after expanding, we get the classical inequality

$$\forall v \in \mathcal{C}, \quad \int_{\Omega} \nabla(u - v) \cdot \nabla u \leq \int_{\Omega} (u - v) f.$$

We can also rewrite the problem as a saddle point problem

Find λ, u such that:

$$\max_{\lambda \in L^2(\Omega), \lambda \geq 0} \min_{u \in H_0^1(\Omega)} \mathcal{L}(u, \lambda) = \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u - \int_{\Omega} f u + \int_{\Omega} \lambda(u - g)^+$$

where $((u - g)^+ = \max(0, u - g))$

This saddle point problem is equivalent to find u, λ such that:

$$\begin{cases} \int_{\Omega} \nabla u \cdot \nabla v + \lambda v^+ d\omega = \int_{\Omega} f u, & \forall v \in H_0^1(\Omega) \\ \int_{\Omega} \mu(u - g)^+ = 0, & \forall \mu \in L^2(\Omega), \mu \geq 0, \lambda \geq 0, \end{cases} \quad (9.54)$$

A algorithm to solve the previous problem is:

1. $k=0$, and choose, λ_0 belong $H^{-1}(\Omega)$
2. loop on $k = 0, \dots$
 - (a) set $\mathcal{I}_k = \{x \in \Omega / \lambda_k + c * (u_{k+1} - g) \leq 0\}$
 - (b) $V_{g,k+1} = \{v \in H_0^1(\Omega) / v = g \text{ on } I_k\}$,
 - (c) $V_{0,k+1} = \{v \in H_0^1(\Omega) / v = 0 \text{ on } I_k\}$,
 - (d) Find $u_{k+1} \in V_{g,k+1}$ and $\lambda_{k+1} \in H^{-1}(\Omega)$ such that

$$\begin{cases} \int_{\Omega} \nabla u_{k+1} \cdot \nabla v_{k+1} d\omega = \int_{\Omega} f v_{k+1}, & \forall v_{k+1} \in V_{0,k+1} \\ < \lambda_{k+1}, v > = \int_{\Omega} \nabla u_{k+1} \cdot \nabla v - f v d\omega \end{cases}$$

where $<, >$ is the duality bracket between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and c is a penalty constant (large enough).

You can find all the mathematic about this algorithm in [33].

Now how to do that in FreeFem++

The full example is:

Example 9.25 (VI.edp)

```

mesh Th=square(20,20);
real eps=1e-5;
fespace Vh(Th,P1); // P1 FE space
int n = Vh.ndof; // number of Degree of freedom
Vh uh,uhp; // solution and previous one
Vh Ik; // to def the set where the containt is reached.
real[int] rhs(n); // to store the right and side of the equation
real c=1000; // the penalty parameter of the algoritm
func f=1; // right hand side function
func fd=0; // Dirichlet boundary condition function
Vh g=0.05; // the discret function g

real[int] Aii(n),Aiin(n); // to store the diagonal of the matrix 2 version

real tgv = 1e30; // a huge value for exact penalization
// of boundary condition

// the variational form of the problem:
varf a(uh,vh) = // definition of the problem
    int2d(Th) ( dx(uh)*dx(vh) + dy(uh)*dy(vh) ) // bilinear form
    - int2d(Th) ( f*vh ) // linear form
    + on(1,2,3,4,uh=fd) ; // boundary condition form

// two version of the matrix of the problem
matrix A=a(Vh,Vh,tgv=tgv,solver=CG); // one changing
matrix AA=a(Vh,Vh,solver=GC); // one for computing residual

// the mass Matrix construction:
varf vM(uh,vh) = int2d(Th) (uh*vh);
matrix M=vM(Vh,Vh); // to do a fast computing of  $L^2$  norm :  $\sqrt{u'*(w=M*u)}$ 

Aii=A.diag; // get the diagonal of the matrix (appear in version 1.46-1)

rhs = a(0,Vh,tgv=tgv);
Ik =0;
uhp=-tgv; // previous value is
Vh lambda=0;
for(int iter=0;iter<100;++iter)
{
    real[int] b(n) ; b=rhs; // get a copy of the Right hand side
    real[int] Ak(n); // the complementary of Ik ( !Ik = (Ik-1))
    // Today the operator Ik- 1. is not implement so we do:
    Ak= 1.; Ak -= Ik[]; // build Ak = ! Ik
    // adding new locking condition on b and on the diagonal if (Ik ==1 )
    b = Ik[] .* g[]; b *= tgv; b -= Ak .* rhs;
    Aiin = Ik[] * tgv; Aiin += Ak .* Aii; // set Aii= tgv i ∈ Ik
    A.diag = Aiin; // set the matrix diagonal (appear in version 1.46-1)
    set(A,solver=CG); // important to change preconditioning for solving
    uh[] = A^-1* b; // solve the problem with more locking condition
    lambda[] = AA * uh[]; // compute the residual ( fast with matrix)
    lambda[] += rhs; // remark rhs = - ∫ f v

    Ik = ( lambda + c*( g- uh)) < 0.; // the new of locking value

```

```

plot(Ik, wait=1, cmm=" lock set ", value=1, ps="VI-lock.eps", fill=1 );
plot(uh, wait=1, cmm="uh", ps="VI-uh.eps");
//      trick to compute L2 norm of the variation (fast method)
real[int] diff(n), Mdiff(n);
diff= uh[]-uhp[];
Mdiff = M*diff;
real err = sqrt(Mdiff'*diff);
cout << "    || u_{k=1} - u_{k} ||_2 " << err << endl;
if(err< eps) break; //      stop test
uhp[]=uh[] ; //      set the previous solution
}
savemesh(Th, "mm", [x,y,uh*10]); //      for medit plotting

```

Remark, as you can see on this example, some vector , or matrix operator are not implemented so a way is to skip the expression and we use operator +=, -= to merge the result.

9.8 Domain decomposition

We present, three classic examples, of domain decomposition technique: first, Schwarz algorithm with overlapping, second Schwarz algorithm without overlapping (also call Shur complement), and last we show to use the conjugate gradient to solve the boundary problem of the Shur complement.

9.8.1 Schwarz Overlap Scheme

To solve

$$-\Delta u = f, \text{ in } \Omega = \Omega_1 \cup \Omega_2 \quad u|_{\Gamma} = 0$$

the Schwarz algorithm runs like this

$$\begin{aligned} -\Delta u_1^{n+1} &= f \text{ in } \Omega_1 & u_1^{n+1}|_{\Gamma_1} &= u_2^n \\ -\Delta u_2^{n+1} &= f \text{ in } \Omega_2 & u_2^{n+1}|_{\Gamma_2} &= u_1^n \end{aligned}$$

where Γ_i is the boundary of Ω_i and on the condition that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and that u_i are zero at iteration 1.

Here we take Ω_1 to be a quadrangle, Ω_2 a disk and we apply the algorithm starting from zero.

Example 9.26 (Schwarz-overlap.edp)

```

int inside = 2; //      inside boundary
int outside = 1; //      outside boundary
border a(t=1,2){x=t;y=0;label=outside;};
border b(t=0,1){x=2;y=t;label=outside;};
border c(t=2,0){x=t ;y=1;label=outside;};
border d(t=1,0){x = 1-t; y = t;label=inside;};
border e(t=0, pi/2){ x= cos(t); y = sin(t);label=inside;};
border e1(t=pi/2, 2*pi){ x= cos(t); y = sin(t);label=outside;};
int n=4;
mesh th = buildmesh( a(5*n) + b(5*n) + c(10*n) + d(5*n));
mesh TH = buildmesh( e(5*n) + e1(25*n) );
plot(th,TH,wait=1); //      to see the 2 meshes

```

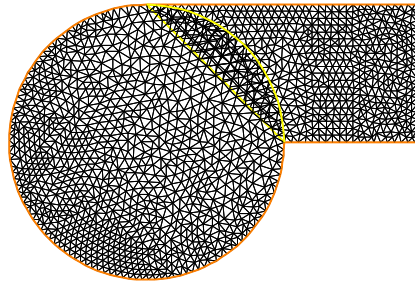


Figure 9.25: The 2 overlapping mesh TH and th

The space and problem definition is :

```

fespace vh(th,P1);
fespace VH(TH,P1);
vh u=0,v; VH U,V;
int i=0;

problem PB(U,V,init=i,solver=Cholesky) =
    int2d(TH) ( dx(U)*dx(V)+dy(U)*dy(V) )
    + int2d(TH) ( -V ) + on(inside,U = u) + on(outside,U= 0 ) ;
problem pb(u,v,init=i,solver=Cholesky) =
    int2d(th) ( dx(u)*dx(v)+dy(u)*dy(v) )
    + int2d(th) ( -v ) + on(inside ,u = U) + on(outside,u = 0 ) ;

```

The calculation loop:

```

for ( i=0 ;i< 10; i++)
{
    PB;
    pb;
    plot (U,u,wait=true);
};

```

9.8.2 Schwarz non Overlap Scheme

To solve

$$-\Delta u = f \text{ in } \Omega = \Omega_1 \cup \Omega_2 \quad u|_{\Gamma} = 0,$$

the Schwarz algorithm for domain decomposition without overlapping runs like this

Let introduce Γ_i is common the boundary of Ω_1 and Ω_2 and $\Gamma_e^i = \partial\Omega_i \setminus \Gamma_i$.

The problem find λ such that $(u_1|_{\Gamma_i} = u_2|_{\Gamma_i})$ where u_i is solution of the following Laplace problem:

$$-\Delta u_i = f \text{ in } \Omega_i \quad u_i|_{\Gamma_i} = \lambda \quad u_i|_{\Gamma_e^i} = 0$$

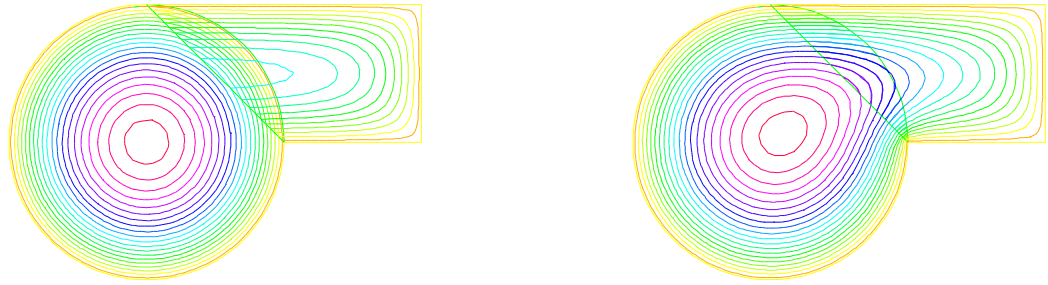


Figure 9.26: Isovalues of the solution at iteration 0 and iteration 9

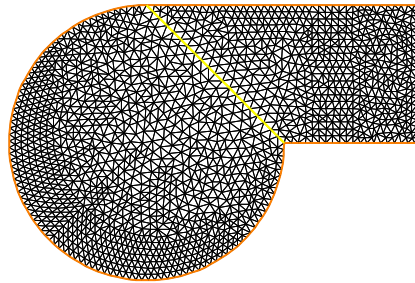


Figure 9.27: The two none overlapping mesh TH and th

To solve this problem we just make a loop with upgrading λ with

$$\lambda = \lambda \pm \frac{(u_1 - u_2)}{2}$$

where the sign $+$ or $-$ of \pm is choose to have convergence.

Example 9.27 (Schwarz-no-overlap.edp)

```

//      schwarz1 without overlapping
int inside = 2;
int outside = 1;
border a(t=1,2){x=t;y=0;label=outside;};
border b(t=0,1){x=2;y=t;label=outside;};
border c(t=2,0){x=t ;y=1;label=outside;};
border d(t=1,0){x = 1-t; y = t;label=inside;};
border e(t=0, 1){ x= 1-t; y = t;label=inside;};
border e1(t=pi/2, 2*pi){ x= cos(t); y = sin(t);label=outside;};
int n=4;
mesh th = buildmesh( a(5*n) + b(5*n) + c(10*n) + d(5*n));
mesh TH = buildmesh ( e(5*n) + e1(25*n) );
```

```

plot(th,TH,wait=1,ps="schwarz-no-u.eps");
fespace vh(th,P1);
fespace VH(TH,P1);
vh u=0,v; VH U,V;
vh lambda=0;
int i=0;

problem PB(U,V,init=i,solver=Cholesky) =
  int2d(TH) ( dx(U)*dx(V)+dy(U)*dy(V) )
+ int2d(TH) ( -V)
+ int1d(TH,inside) (lambda*V) + on(outside,U= 0 ) ;
problem pb(u,v,init=i,solver=Cholesky) =
  int2d(th) ( dx(u)*dx(v)+dy(u)*dy(v) )
+ int2d(th) ( -v)
+ int1d(th,inside) (-lambda*v) + on(outside,u = 0 ) ;

for ( i=0 ;i< 10; i++)
{
  PB;
  pb;
  lambda = lambda - (u-U)/2;
  plot(U,u,wait=true);
};

plot(U,u,ps="schwarz-no-u.eps");

```

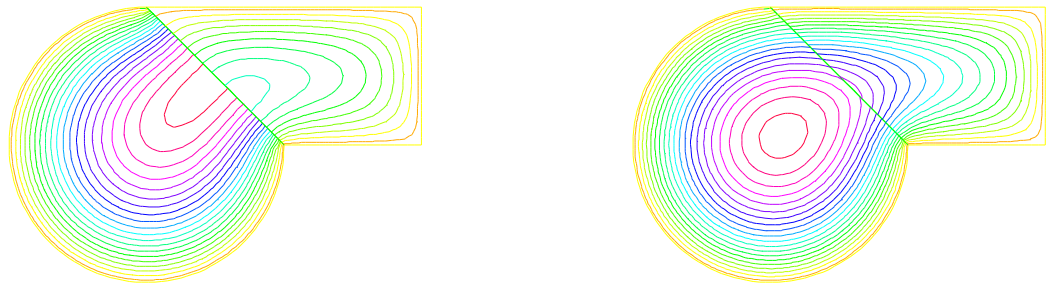


Figure 9.28: Isovalues of the solution at iteration 0 and iteration 9 without overlapping

9.8.3 Schwarz-gc.edp

To solve

$$-\Delta u = f \text{ in } \Omega = \Omega_1 \cup \Omega_2 \quad u|_{\Gamma} = 0,$$

the Schwarz algorithm for domain decomposition without overlapping runs like this
 Let introduce Γ_i is common the boundary of Ω_1 and Ω_2 and $\Gamma_e^i = \partial\Omega_i \setminus \Gamma_i$.

The problem find λ such that $(u_1|_{\Gamma_i} = u_2|_{\Gamma_i})$ where u_i is solution of the following Laplace problem:

$$-\Delta u_i = f \text{ in } \Omega_i \quad u_i|_{\Gamma_i} = \lambda \quad u_i|_{\Gamma_e} = 0$$

The version of this example for Shur component. The border problem is solve with conjugate gradient.

First, we construct the two domain

Example 9.28 (Schwarz-gc.edp)

```
//      Schwarz without overlapping (Shur complement Neumann -> Dirichet)
real cpu=clock();
int inside = 2;
int outside = 1;

border Gamma1(t=1,2){x=t;y=0;label=outside;};
border Gamma2(t=0,1){x=2;y=t;label=outside;};
border Gamma3(t=2,0){x=t ;y=1;label=outside;};

border GammaInside(t=1,0){x = 1-t; y = t;label=inside;};

border GammaArc(t=pi/2, 2*pi){ x= cos(t); y = sin(t);label=outside;};
int n=4;

//      build the mesh of  $\Omega_1$  and  $\Omega_2$ 
mesh Th1 = buildmesh( Gamma1(5*n) + Gamma2(5*n) + GammaInside(5*n) + Gamma3(5*n));
mesh Th2 = buildmesh ( GammaInside(-5*n) + GammaArc(25*n) );
plot (Th1,Th2);

//      defined the 2 FE space
fespace Vh1 (Th1,P1),          Vh2 (Th2,P1);
```

Note 9.6 It is impossible to define a function just on a part of boundary, so the λ function must be defined on the all domain Ω_1 such as

```
Vh1 lambda=0; //      take  $\lambda \in V_{h1}$ 
```

The two Poisson problem:

```
Vh1 u1,v1;          Vh2 u2,v2;
int i=0; //      for factorization optimization
problem Pb2(u2,v2,init=i,solver=Cholesky) =
  int2d(Th2) ( dx(u2)*dx(v2)+dy(u2)*dy(v2) )
+ int2d(Th2) ( -v2)
+ int1d(Th2,inside) (-lambda*v2) + on(outside,u2= 0 ) ;
problem Pb1(u1,v1,init=i,solver=Cholesky) =
  int2d(Th1) ( dx(u1)*dx(v1)+dy(u1)*dy(v1) )
+ int2d(Th1) ( -v1)
+ int1d(Th1,inside) (+lambda*v1) + on(outside,u1 = 0 ) ;
```

or, we define a border matrix , because the λ function is none zero inside the domain Ω_1 :

```
varf b(u2,v2,solver=CG) =int1d(Th1,inside) (u2*v2);
matrix B= b(Vh1,Vh1,solver=CG);
```

The boundary problem function,

$$\lambda \longrightarrow \int_{\Gamma_i} (u_1 - u_2) v_1$$

```
func real[int] BoundaryProblem(real[int] &l)
{
    lambda[] = l; // make FE function form l
    Pb1; Pb2;
    i++; // no refactorization i != 0
    v1 = -(u1 - u2);
    lambda[] = B * v1[];
    return lambda[];
};
```

Note 9.7 The difference between the two notations v_1 and $v_1[]$ is: v_1 is the finite element function and $v_1[]$ is the vector in the canonical basis of the finite element function v_1 .

```
Vh1 p=0, q=0; // solve the problem with Conjugate Gradient
LinearCG(BoundaryProblem, p[], eps=1.e-6, nbiter=100);
// compute the final solution, because CG works with increment
BoundaryProblem(p[]); // solve again to have right u1, u2

cout << " -- CPU time schwarz-gc:" << clock() - cpu << endl;
plot(u1, u2); // plot
```

9.9 Fluid/Structures Coupled Problem

This problem involves the Lamé system of elasticity and the Stokes system for viscous fluids with velocity \mathbf{u} and pressure p :

$$-\Delta \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{u}_\Gamma \quad \text{on } \Gamma = \partial\Omega$$

where \mathbf{u}_Γ is the velocity of the boundaries. The force that the fluid applies to the boundaries is the normal stress

$$\mathbf{h} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \mathbf{n} - p \mathbf{n}$$

Elastic solids subject to forces deform: a point in the solid, at (x, y) goes to (X, Y) after. When the displacement vector $\mathbf{v} = (v_1, v_2) = (X - x, Y - y)$ is small, Hooke's law relates the stress tensor σ inside the solid to the deformation tensor ϵ :

$$\sigma_{ij} = \lambda \delta_{ij} \nabla \cdot \mathbf{v} + 2\mu \epsilon_{ij}, \quad \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

where δ is the Kronecker symbol and where λ, μ are two constants describing the material mechanical properties in terms of the modulus of elasticity, and Young's modulus.

The equations of elasticity are naturally written in variational form for the displacement vector $\mathbf{v}(x) \in V$ as

$$\int_{\Omega} [2\mu \epsilon_{ij}(\mathbf{v}) \epsilon_{ij}(\mathbf{w}) + \lambda \epsilon_{ii}(\mathbf{v}) \epsilon_{jj}(\mathbf{w})] = \int_{\Omega} \mathbf{g} \cdot \mathbf{w} + \int_{\Gamma} \mathbf{h} \cdot \mathbf{w}, \quad \forall \mathbf{w} \in V$$

The data are the gravity force \mathbf{g} and the boundary stress \mathbf{h} .

Example 9.29 (fluidStruct.edp) In our example the Lamé system and the Stokes system are coupled by a common boundary on which the fluid stress creates a displacement of the boundary and hence changes the shape of the domain where the Stokes problem is integrated. The geometry is that of a vertical driven cavity with an elastic lid. The lid is a beam with weight so it will be deformed by its own weight and by the normal stress due to the fluid reaction. The cavity is the 10×10 square and the lid is a rectangle of height $l = 2$.

A beam sits on a box full of fluid rotating because the left vertical side has velocity one. The beam is bent by its own weight, but the pressure of the fluid modifies the bending. The bending displacement of the beam is given by (uu, vv) whose solution is given as follows.

```
//      Fluid-structure interaction for a weighting beam sitting on a
//      square cavity filled with a fluid.

int bottombeam = 2; //      label of bottombeam
border a(t=2,0) { x=0; y=t ;label=1;}; //      left beam
border b(t=0,10) { x=t; y=0 ;label=bottombeam;}; //      bottom of beam
border c(t=0,2) { x=10; y=t ;label=1;}; //      righth beam
border d(t=0,10) { x=10-t; y=2; label=3;}; //      top beam
real E = 21.5;
real sigma = 0.29;
real mu = E/(2*(1+sigma));
real lambda = E*sigma/((1+sigma)*(1-2*sigma));
real gravity = -0.05;
mesh th = buildmesh( b(20)+c(5)+d(20)+a(5));
fespace Vh(th,P1);
Vh uu,w,vv,s,fluidforce=0;
cout << "lambda,mu,gravity ="<<lambda<< " " << mu << " " << gravity << endl;
//      deformation of a beam under its own weight
solve bb([uu,vv],[w,s]) =
    int2d(th) (
        lambda*div(w,s)*div(uu,vv)
        +2.*mu*( epsilon(w,s)'*epsilon(uu,vv) )
    )
    + int2d(th) (-gravity*s)
    + on(1,uu=0,vv=0)
    + fluidforce[];
;

plot([uu,vv],wait=1);
mesh th1 = movemesh(th, [x+uu, y+vv]);
plot(th1,wait=1);
```

Then Stokes equation for fluids at low speed are solved in the box below the beam, but the beam has deformed the box (see border h):

```
//      Stokes on square b,e,f,g driven cavite on left side g
border e(t=0,10) { x=t; y=-10; label= 1; }; //      bottom
border f(t=0,10) { x=10; y=-10+t ; label= 1; }; //      right
border g(t=0,10) { x=0; y=-t ;label= 2;}; //      left
border h(t=0,10) { x=t; y=vv(t,0)*( t>=0.001 )*( t <= 9.999);
    label=3;}; //      top of cavity deformed

mesh sh = buildmesh(h(-20)+f(10)+e(10)+g(10));
plot(sh,wait=1);
```

We use the Uzawa conjugate gradient to solve the Stokes problem like in example Section 9.6.2

```

fespace Xh (sh,P2), Mh (sh,P1);
Xh u1,u2,v1,v2;
Mh p,q,ppp;

varf bx(u1,q) = int2d(sh) ( -(dx(u1)*q));

varf by(u1,q) = int2d(sh) ( -(dy(u1)*q));

varf Lap(u1,u2)= int2d(sh) ( dx(u1)*dx(u2) + dy(u1)*dy(u2) )
                  + on(2,u1=1) + on(1,3,u1=0) ;

Xh bcl; bcl[] = Lap(0,Xh);
Xh brhs;

matrix A= Lap(Xh,Xh,solver=CG);
matrix Bx= bx(Xh,Mh);
matrix By= by(Xh,Mh);
Xh bcx=0,bcy=1;

```

```

func real[int] divup(real[int] & pp)
{
  int verb=verbosity;
  verbosity=0;
  brhs[] = Bx'*pp; brhs[] += bcl[] .*bcx[];
  u1[] = A^-1*brhs[];
  brhs[] = By'*pp; brhs[] += bcl[] .*bcy[];
  u2[] = A^-1*brhs[];
  ppp[] = Bx*u1[];
  ppp[] += By*u2[];
  verbosity=verb;
  return ppp[] ;
};

```

do a loop on the two problem

```

for(step=0;step<2;++step)
{
  p=0;q=0;u1=0;v1=0;

  LinearCG(divup,p[],eps=1.e-3,nbiter=50);
  divup(p[]);

```

Now the beam will feel the stress constraint from the fluid:

```

Vh sigma11,sigma22,sigma12;
Vh uu1=uu,vv1=vv;

sigma11([x+uu,y+vv]) = (2*dx(u1)-p);
sigma22([x+uu,y+vv]) = (2*dy(u2)-p);
sigma12([x+uu,y+vv]) = (dx(u1)+dy(u2));

```

which comes as a boundary condition to the PDE of the beam:

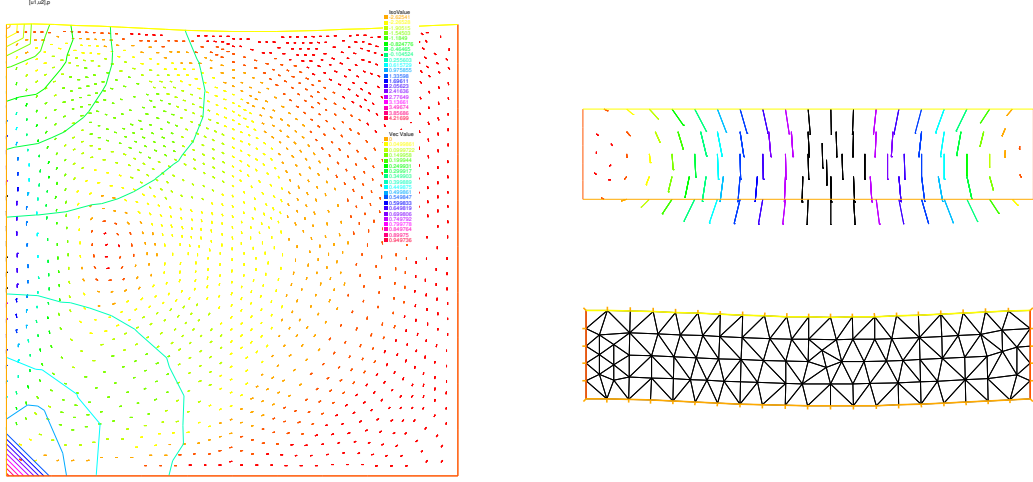


Figure 9.29: Fluid velocity and pressure (left) and displacement vector (center) of the structure and displaced geometry (right) in the fluid-structure interaction of a soft side and a driven cavity

```

solve  bbst([uu,vv],[w,s],init=i)  =
      int2d(th) (
          lambda*div(w,s)*div(uu,vv)
          +2.*mu*( epsilon(w,s)'*epsilon(uu,vv) )
      )
+ int2d(th) (-gravity*s)
+ int1d(th,bottombeam) ( -coef*(      sigma11*N.x*w + sigma22*N.y*s
                                     + sigma12*(N.y*w+N.x*s) ) )
+ on(1,uu=0,vv=0);
plot([uu,vv],wait=1);
real  err = sqrt(int2d(th) ( (uu-uul)^2 + (vv-vv1)^2 ));
cout << " Erreur L2 = " << err << "-----\n";

```

Notice that the matrix generated by *bbst* is reused (see *init=i*). Finally we deform the beam

```

th1 = movemesh(th, [x+0.2*uu, y+0.2*vv]);
plot(th1,wait=1);
} //      end of loop

```

9.10 Transmission Problem

Consider an elastic plate whose displacement change vertically, which is made up of three plates of different materials, welded on each other. Let Ω_i , $i = 1, 2, 3$ be the domain occupied by i -th material with tension μ_i (see Section 9.1.1). The computational domain Ω is the interior of $\overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$. The vertical displacement $u(x, y)$ is obtained from

$$-\mu_i \Delta u = f \text{ in } \Omega_i \quad (9.55)$$

$$\mu_i \partial_n u|_{\Gamma_i} = -\mu_j \partial_n u|_{\Gamma_j} \text{ on } \overline{\Omega_i} \cap \overline{\Omega_j} \quad \text{if } 1 \leq i < j \leq 3 \quad (9.56)$$

where $\partial_n u|_{\Gamma_i}$ denotes the value of the normal derivative $\partial_n u$ on the boundary Γ_i of the domain Ω_i . By introducing the characteristic function χ_i of Ω_i , that is,

$$\chi_i(x) = 1 \quad \text{if } x \in \Omega_i; \quad \chi_i(x) = 0 \quad \text{if } x \notin \Omega_i \quad (9.57)$$

we can easily rewrite (9.55) and (9.56) to the weak form. Here we assume that $u = 0$ on $\Gamma = \partial\Omega$.
 problem Transmission: For a given function f , find u such that

$$a(u, v) = \ell(f, v) \quad \text{for all } v \in H_0^1(\Omega) \quad (9.58)$$

$$a(u, v) = \int_{\Omega} \mu \nabla u \cdot \nabla v, \quad \ell(f, v) = \int_{\Omega} f v$$

where $\mu = \mu_1 \chi_1 + \mu_2 \chi_2 + \mu_3 \chi_3$. Here we notice that μ become the discontinuous function.

With dissipation, and at the thermal equilibrium, the temperature equation is:

This example explains the definition and manipulation of *region*, i.e. subdomains of the whole domain.

Consider this L-shaped domain with 3 diagonals as internal boundaries, defining 4 subdomains:

```

//      example using region keyword
//      construct a mesh with 4 regions (sub-domains)
border a(t=0,1){x=t;y=0;};
border b(t=0,0.5){x=1;y=t;};
border c(t=0,0.5){x=1-t;y=0.5;};
border d(t=0.5,1){x=0.5;y=t;};
border e(t=0.5,1){x=1-t;y=1;};
border f(t=0,1){x=0;y=1-t;};

//      internal boundary
border i1(t=0,0.5){x=t;y=1-t;};
border i2(t=0,0.5){x=t;y=t;};
border i3(t=0,0.5){x=1-t;y=t;};

mesh th = buildmesh (a(6) + b(4) + c(4) +d(4) + e(4) +
    f(6)+i1(6)+i2(6)+i3(6));
fespace Ph(th,P0); //      constant discontinuous functions / element
fespace Vh(th,P1); //      P1 continuous functions / element

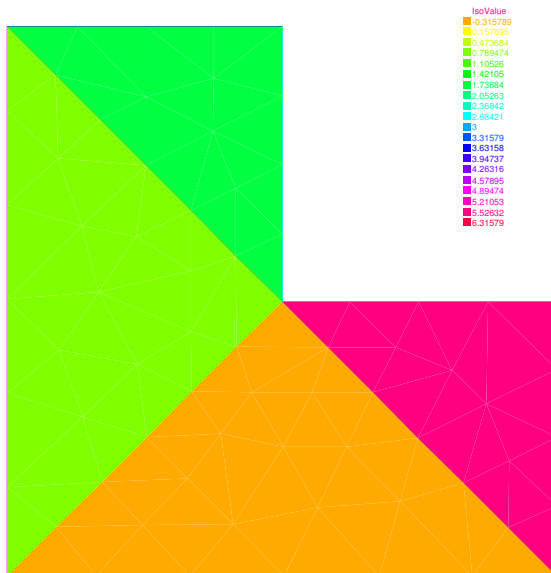
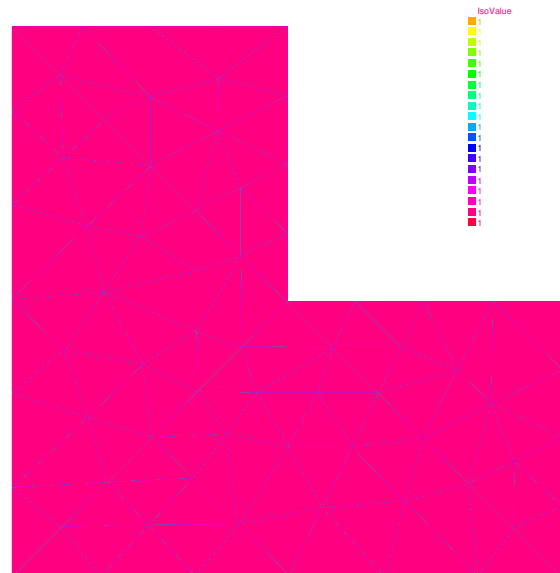
Ph reg=region; //      defined the P0 function associated to region number
plot(reg,fill=1,wait=1,value=1);
```

`region` is a keyword of FreeFem++ which is in fact a variable depending of the current position (is not a function today, use `Ph reg=region;` to set a function). This variable value returned is the number of the subdomain of the current position. This number is defined by "buildmesh" which scans while building the mesh all its connected component. So to get the number of a region containing a particular point one does:

```

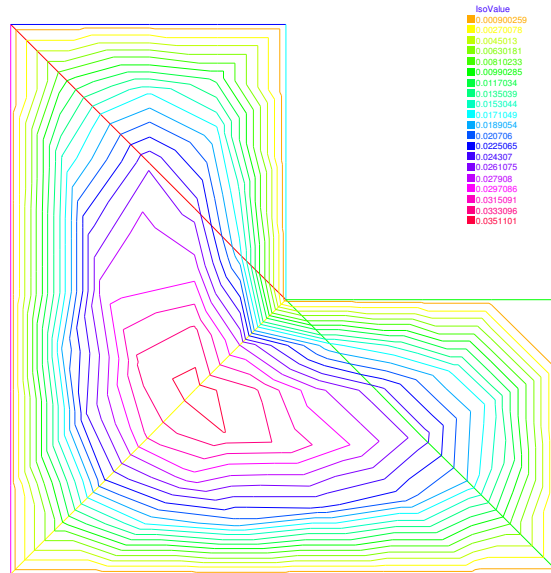
int nupper=reg(0.4,0.9); //      get the region number of point (0.4,0.9)
int nlower=reg(0.9,0.1); //      get the region number of point (0.4,0.1)
cout << " nlower " << nlower << ", nupper = " << nupper<< endl;
//      defined the characteristics functions of upper and lower region
Ph nu=1+5*(region==nlower) + 10*(region==nupper);
plot(nu,fill=1,wait=1);
```

This is particularly useful to define discontinuous functions such as might occur when one part of the domain is copper and the other one is iron, for example.

Figure 9.30: the function reg Figure 9.31: the function nu

We this in mind we proceed to solve a Laplace equation with discontinuous coefficients (ν is 1, 6 and 11 below).

```
Ph nu=1+5*(region==nlower) + 10*(region==nupper);
plot(nu,fill=1,wait=1);
problem lap(u,v) = int2d(th) ( nu*( dx(u)*dx(v)*dy(u)*dy(v) ))
                  + int2d(-1*v) + on(a,b,c,d,e,f,u=0);
plot(u);
```

Figure 9.32: the isovalue of the solution u

9.11 Free Boundary Problem

The domain Ω is defined with:

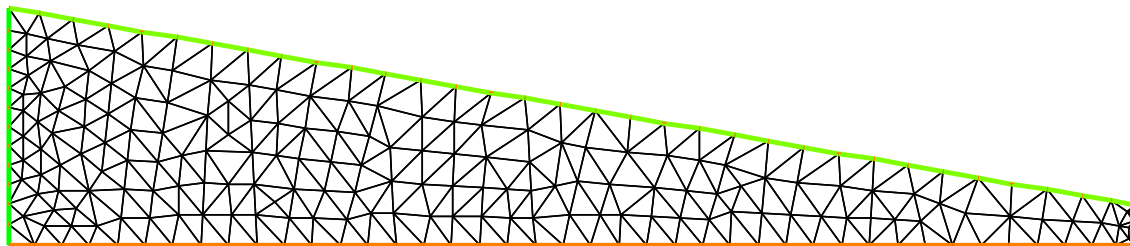
```

real L=10;                                // longueur du domaine
real h=2.1;                                // hauteur du bord gauche
real h1=0.35;                              // hauteur du bord droite

// maillage d'un trapèze
border a (t=0,L) {x=t;y=0;};               // bottom:  $\Gamma_a$ 
border b (t=0,h1) {x=L;y=t;};              // right:  $\Gamma_b$ 
border f (t=L,0) {x=t;y=t*(h1-h)/L+h;};    // free surface:  $\Gamma_f$ 
border d (t=h,0) {x=0;y=t;};               // left:  $\Gamma_d$ 

int n=4;
mesh Th=buildmesh (a(10*n)+b(6*n)+f(8*n)+d(3*n));
plot (Th,ps="dTh.eps");

```

Figure 9.33: The mesh of the domain Ω

The free boundary problem is:

Find u and Ω such that:

$$\left\{ \begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ u = y & \text{on } \Gamma_b \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_d \cup \Gamma_a \\ \frac{\partial u}{\partial n} = \frac{q}{K} n_x \text{ and } u = y & \text{on } \Gamma_f \end{array} \right.$$

We use a fixed point method; $\Omega^0 = \Omega$
in two step, first we solve the classical following problem:

$$\left\{ \begin{array}{ll} -\Delta u = 0 & \text{in } \Omega^n \\ u = y & \text{on } \Gamma_b^n \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_d^n \cup \Gamma_a^n \\ u = y & \text{on } \Gamma_f^n \end{array} \right.$$

The variational formulation is:

find u on $V = H^1(\Omega^n)$, such that $u = y$ on Γ_b^n and Γ_f^n

$$\int_{\Omega^n} \nabla u \nabla u' = 0, \quad \forall u' \in V \text{ with } u' = 0 \text{ on } \Gamma_b^n \cup \Gamma_f^n$$

and secondly to construct a domain deformation $\mathcal{F}(x, y) = [x, y - v(x, y)]$
where v is solution of the following problem:

$$\left\{ \begin{array}{ll} -\Delta v = 0 & \text{in } \Omega^n \\ v = 0 & \text{on } \Gamma_a^n \\ \frac{\partial v}{\partial n} = 0 & \text{on } \Gamma_b^n \cup \Gamma_d^n \\ \frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} - \frac{q}{K} n_x & \text{on } \Gamma_f^n \end{array} \right.$$

The variational formulation is:

find v on V , such that $v = 0$ on Γ_a^n

$$\int_{\Omega^n} \nabla v \nabla v' = \int_{\Gamma_f^n} \left(\frac{\partial u}{\partial n} - \frac{q}{K} n_x \right) v', \quad \forall v' \in V \text{ with } v' = 0 \text{ on } \Gamma_a^n$$

Finally the new domain $\Omega^{n+1} = \mathcal{F}(\Omega^n)$

Example 9.30 (freeboundary.edp) *The FreeFem++ implementation is:*

```

real q=0.02; // flux entrant
real K=0.5; // permeabilité

fespace Vh(Th,P1);
int j=0;

Vh u,v,uu,vv;

problem Pu(u,uu,solver=CG) = int2d(Th) ( dx(u)*dx(uu)+dy(u)*dy(uu) )
+ on(b,f,u=y) ;

problem Pv(v,vv,solver=CG) = int2d(Th) ( dx(v)*dx(vv)+dy(v)*dy(vv) )
+ on (a, v=0) + int1d(Th,f) (vv*((q/K)*N.y- (dx(u)*N.x+dy(u)*N.y))) ;

```

```

real errv=1;
real erradap=0.001;
verbosity=1;
while(errv>1e-6)
{
    j++;
    Pu;
    Pv;
    plot (Th,u,v ,wait=0);
    errv=int1d(Th,f) (v*v);
    real coef=1;

                                                                    //
    real mintcc = checkmovemesh (Th, [x,y])/5.;
    real mint = checkmovemesh (Th, [x,y-v*coef]);

    if (mint<mintcc || j%10==0) {                                     // mesh to bad => remeshing
        Th=adaptmesh (Th,u,err=erradap ) ;
        mintcc = checkmovemesh (Th, [x,y])/5.;
    }

    while (1)
    {
        real mint = checkmovemesh (Th, [x,y-v*coef]);

        if (mint>mintcc) break;

        cout << " min |T| " << mint << endl;
        coef /= 1.5;
    }

    Th=movemesh (Th, [x,y-coef*v]);                                     // calcul de la deformation
    cout << "\n\n"<<j <<"----- errv = " << errv << "\n\n";

}
plot (Th,ps="d_Thf.eps");
plot (u,wait=1,ps="d_u.eps");

```

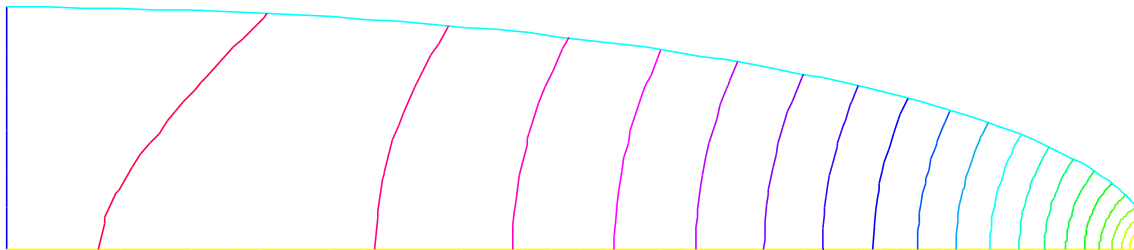
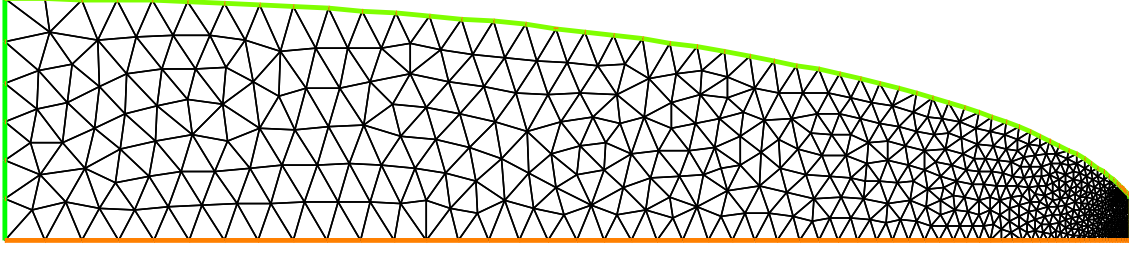


Figure 9.34: The final solution on the new domain Ω^{72}

Figure 9.35: The adapted mesh of the domain Ω^{72}

9.12 Non linear Elasticity (nolinear-elas.edp)

The nonlinear elasticity problem is: find the displacement (u_1, u_2) minimizing J

$$\min J(u_1, u_2) = \int_{\Omega} f(F2) - \int_{\Gamma_p} P_a u_2$$

where $F2(u_1, u_2) = A(E[u_1, u_2], E[u_1, u_2])$ and $A(X, Y)$ is bilinear sym. positive form with respect two matrix X, Y . where f is a given \mathcal{C}^2 function, and $E[u_1, u_2] = (E_{ij})_{i=1,2, j=1,2}$ is the Green-Saint Venant deformation tensor defined with:

$$E_{ij} = 0.5((\partial_i u_j + \partial_j u_i) + \sum_k \partial_i u_k \times \partial_j u_k)$$

Denote $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2)$.

So, the differential of J is

$$DJ(\mathbf{u})(\mathbf{v}) = \int DF2(\mathbf{u})(\mathbf{v}) f'(F2(\mathbf{u})) - \int_{\Gamma_p} P_a v_2$$

where $DF2(\mathbf{u})(\mathbf{v}) = 2 A(DE[\mathbf{u}](\mathbf{v}), E[\mathbf{u}])$ and DE is the first differential of E .

The second order differential is

$$\begin{aligned} D^2 J(\mathbf{u})(\mathbf{v}, \mathbf{w}) &= \int DF2(\mathbf{u})(\mathbf{v}) DF2(\mathbf{u})(\mathbf{w}) f''(F2(\mathbf{u})) \\ &+ \int D^2 F2(\mathbf{u})(\mathbf{v}, \mathbf{w}) f'(F2(\mathbf{u})) \end{aligned}$$

where

$$D^2 F2(\mathbf{u})(\mathbf{v}, \mathbf{w}) = 2 A(D^2 E[\mathbf{u}](\mathbf{v}, \mathbf{w}), E[\mathbf{u}]) + 2 A(DE[\mathbf{u}](\mathbf{v}), DE[\mathbf{u}](\mathbf{w})).$$

and $D^2 E$ is the second differential of E .

So all notations can be define with macros:

macro EL (u, v) [dx (u) , (dx (v) + dy (u)) , dy (v)] // is $[\epsilon_{11}, 2\epsilon_{12}, \epsilon_{22}]$

macro ENL (u, v) [(dx (u) * dx (u) + dx (v) * dx (v)) * 0.5 , (dx (u) * dy (u) + dx (v) * dy (v)) , (dy (u) * dy (u) + dy (v) * dy (v)) * 0.5] // EOM ENL

macro dENL (u, v, uu, vv) [(dx (u) * dx (uu) + dx (v) * dx (vv)) , (dx (u) * dy (uu) + dx (v) * dy (vv) + dx (uu) * dy (u) + dx (vv) * dy (v)) ,

```

(dy(u)*dy(uu)+dy(v)*dy(vv)) ]                                     //

macro E(u,v) (EL(u,v)+ENL(u,v))                                     //      is [E11,2E12,E22]
macro dE(u,v,uu,vv) (EL(uu,vv)+dENL(u,v,uu,vv))                   //
macro ddE(u,v,uu,vv,uuu,vvv) dENL(uuu,vvv,uu,vv)                  //
macro F2(u,v) (E(u,v)'*A*E(u,v))                                    //
macro dF2(u,v,uu,vv) (E(u,v)'*A*dE(u,v,uu,vv)*2.)                 //
macro ddF2(u,v,uu,vv,uuu,vvv) (
    (dE(u,v,uu,vv)'*A*dE(u,v,uuu,vvv))*2.
    + (E(u,v)'*A*ddE(u,v,uu,vv,uuu,vvv))*2.)                       //      EOM

```

The Newton Method is

choose $n = 0$, and u_O, v_O the initial displacement

- loop:
- find (du, dv) : solution of
$$D^2 J(u_n, v_n)((w, s), (du, dv)) = DJ(u_n, v_n)(w, s), \quad \forall w, s$$
- $un = un - du, \quad vn = vn - dv$
- until (du, dv) small is enough

The way to implement this algorithm in FreeFem++ is use a macro tool to implement A and $F2, f, f', f''$.

A macro is like in `ccp` preprocessor of C++ , but this begin by `macro` and the end of the macro definition is before the comment `//`. In this case the macro is very useful because the type of parameter can be change. And it is easy to make automatic differentiation.

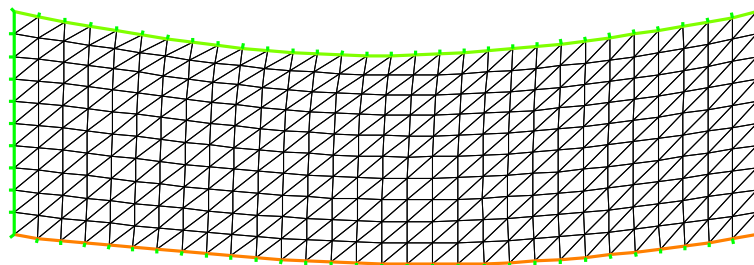


Figure 9.36: The deformed domain

```

//      non linear elasticity model

//      for hyper elasticity problem
//      -----

macro f(u) (u)                                                       //      end of macro
macro df(u) (1)                                                       //      end of macro
macro ddf(u) (0)                                                       //      end of macro

//      -- du caouchouc --- (see the notes of Herve Le Dret.)
//      -----

```



```

varf vmass([uu,vv],[w,s],solver=CG) = int2d(Th)( uu*w + vv*s );
matrix M=vmass(Vh,Vh);
problem NonLin([uu,vv],[w,s],solver=LU)=
  int2d(Th,qforder=1) ( //       $(D^2J(un))$  part
                        dF2(un,vn,uu,vv)*dF2(un,vn,w,s)*ddfe2
                        + ddF2(un,vn,w,s,uu,vv)*dfe2
                      )
  - int1d(Th,3) (Pa*s)
  - int2d(Th,qforder=1) ( //       $(DJ(un))$  part
                        dF2(un,vn,w,s)*dfe2 )
  + on(right,left,uu=0,vv=0);
;

//      Newton's method
//      -----

Sh u1,v1;
for (int i=0;i<10;i++)
{
  cout << "Loop " << i << endl;
  e2 = F2(un,vn);
  dfe2 = df(e2);
  ddfe2 = ddf(e2);
  cout << "  e2 max " << e2[].max << " , min" << e2[].min << endl;
  cout << " de2 max " << dfe2[].max << " , min" << dfe2[].min << endl;
  cout << " dde2 max " << ddfe2[].max << " , min" << ddfe2[].min << endl;
  NonLin; //      compute  $[uu,vv] = (D^2J(un))^{-1}(DJ(un))$ 

  w[] = M*uu[];
  real res = sqrt(w[]' * uu[]); //      norme  $L^2$  of  $[uu,vv]$ 
  u1 = uu;
  v1 = vv;
  cout << " L^2 residual = " << res << endl;
  cout << " u1 min =" << u1[].min << " , u1 max= " << u1[].max << endl;
  cout << " v1 min =" << v1[].min << " , v2 max= " << v1[].max << endl;
  plot([uu,vv],wait=1,cmm=" uu, vv " );
  un[] -= uu[];
  plot([un,vn],wait=1,cmm=" displacement " );
  if (res<1e-5) break;
}

plot([un,vn],wait=1);
mesh th1 = movemesh(Th, [x+un, y+vn]);
plot(th1,wait=1); //      see figure 9.36

```

9.13 Compressible Neo-Hookean Materials: Computational Solutions

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9.13.1 Notation

In what follows, the symbols \mathbf{u} , \mathbf{F} , \mathbf{B} , \mathbf{C} , $\underline{\sigma}$ denote, respectively, the displacement field, the deformation gradient, the left Cauchy-Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, the right Cauchy-Green strain tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and the Cauchy stress tensor. We also introduce the symbols $I_1 := \text{tr } \mathbf{C}$ and $J := \det \mathbf{F}$. Use will be made of the identity

$$\frac{\partial J}{\partial \mathbf{C}} = J\mathbf{C}^{-1} \quad (9.59)$$

The symbol \mathbf{I} denotes the identity tensor. The symbol Ω_0 denotes the reference configuration of the body to be deformed. The unit volume in the reference (resp., deformed) configuration is denoted dV (resp., dV_0); these two are related by

$$dV = JdV_0,$$

which allows an integral over Ω involving the Cauchy stress \mathbf{T} to be rewritten as an integral of the Kirchhoff stress $\kappa = J\mathbf{T}$ over Ω_0 .

Recommended References

For an exposition of nonlinear elasticity and of the underlying linear- and tensor algebra, see [34]. For an advanced mathematical analysis of the Finite Element Method, see [35]. An explanation of the Finite Element formulation of a nonlinear elastostatic boundary value problem, see <http://www.engin.brown.edu/courses/en222/Notes/FEMfinitestrain/FEMfinitestrain.htm>.

9.13.2 A Neo-Hookean Compressible Material

Constitutive Theory and Tangent Stress Measures The strain energy density function is given by

$$W = \frac{\mu}{2}(I_1 - \text{tr } \mathbf{I} - 2 \ln J) \quad (9.60)$$

(see [32], formula (12)).

The corresponding 2nd Piola-Kirchhoff stress tensor is given by

$$\mathbf{S}_n := \frac{\partial W}{\partial \mathbf{E}}(\mathbf{F}_n) = \mu(\mathbf{I} - \mathbf{C}^{-1}) \quad (9.61)$$

The Kirchhoff stress, then, is

$$\kappa = \mathbf{F}\mathbf{S}\mathbf{F}^T = \mu(\mathbf{B} - \mathbf{I}) \quad (9.62)$$

The tangent Kirchhoff stress tensor at \mathbf{F}_n acting on $\delta\mathbf{F}_{n+1}$ is, consequently,

$$\frac{\partial \kappa}{\partial \mathbf{F}}(\mathbf{F}_n)\delta\mathbf{F}_{n+1} = \mu [\mathbf{F}_n(\delta\mathbf{F}_{n+1})^T + \delta\mathbf{F}_{n+1}(\mathbf{F}_n)^T] \quad (9.63)$$

The Weak Form of the BVP in the Absence of Body (External) Forces The Ω_0 we are considering is an elliptical annulus, whose boundary consists of two concentric ellipses (each allowed to be a circle as a special case), with the major axes parallel. Let P denote the dead stress load (traction) on a portion $\partial\Omega_0^t$ (= the inner ellipse) of the boundary $\partial\Omega_0$. On the rest of the boundary, we prescribe zero displacement.

The weak formulation of the boundary value problem is

$$0 = \int_{\Omega_0} \kappa[\mathbf{F}] : \{(\nabla \otimes \mathbf{w})(\mathbf{F})^{-1}\} - \int_{\partial\Omega_0^t} P \cdot \hat{N}_0 \quad \Bigg\}$$

For brevity, in the rest of this section we assume $P = 0$. The provided `FreeFem++` code, however, does not rely on this assumption and allows for a general value and direction of P .

Given a Newton approximation \mathbf{u}_n of the displacement field \mathbf{u} satisfying the BVP, we seek the correction $\delta\mathbf{u}_{n+1}$ to obtain a better approximation

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \delta\mathbf{u}_{n+1}$$

by solving the weak formulation

$$\left. \begin{aligned} 0 &= \int_{\Omega_0} \kappa[\mathbf{F}_n + \delta\mathbf{F}_{n+1}] : \{(\nabla \otimes \mathbf{w})(\mathbf{F}_n + \delta\mathbf{F}_{n+1})^{-1}\} - \int_{\partial\Omega_0} P \cdot \hat{N}_0 \\ &= \int_{\Omega_0} \left\{ \kappa[\mathbf{F}_n] + \frac{\partial\kappa}{\partial\mathbf{F}}[\mathbf{F}_n] \delta\mathbf{F}_{n+1} \right\} : \{(\nabla \otimes \mathbf{w})(\mathbf{F}_n + \delta\mathbf{F}_{n+1})^{-1}\} \\ &= \int_{\Omega_0} \left\{ \kappa[\mathbf{F}_n] + \frac{\partial\kappa}{\partial\mathbf{F}}[\mathbf{F}_n] \delta\mathbf{F}_{n+1} \right\} : \{(\nabla \otimes \mathbf{w})(\mathbf{F}_n^{-1} + \mathbf{F}_n^{-2} \delta\mathbf{F}_{n+1})\} \\ &= \int_{\Omega_0} \kappa[\mathbf{F}_n] : \{(\nabla \otimes \mathbf{w})\mathbf{F}_n^{-1}\} \\ &\quad - \int_{\Omega_0} \kappa[\mathbf{F}_n] : \{(\nabla \otimes \mathbf{w})(\mathbf{F}_n^{-2} \delta\mathbf{F}_{n+1})\} \\ &\quad + \int_{\Omega_0} \left\{ \frac{\partial\kappa}{\partial\mathbf{F}}[\mathbf{F}_n] \delta\mathbf{F}_{n+1} \right\} : \{(\nabla \otimes \mathbf{w})\mathbf{F}_n^{-1}\} \end{aligned} \right\} \quad \text{for all test functions } \mathbf{w}, \quad (9.64)$$

where we have taken

$$\delta\mathbf{F}_{n+1} = \nabla \otimes \delta\mathbf{u}_{n+1}$$

Note: Contrary to standard notational use, the symbol δ here bears no variational context. By δ we mean simply an increment in the sense of Newton's Method. The role of a variational virtual displacement here is played by \mathbf{w} .

9.13.3 An Approach to Implementation in `FreeFem++`

The associated file is `examples++-tutorial/nl-elast-neo-Hookean.edp`.

Introducing the code-like notation, where a string in $\langle \rangle$'s is to be read as one symbol, the individual components of the tensor

$$\langle TanK \rangle := \frac{\partial\kappa}{\partial\mathbf{F}}[\mathbf{F}_n] \delta\mathbf{F}_{n+1} \quad (9.65)$$

will be implemented as the macros $\langle TanK11 \rangle, \langle TanK12 \rangle, \dots$

The individual components of the tensor quantities

$$\mathbf{D}_1 := \mathbf{F}_n(\delta\mathbf{F}_{n+1})^T + \delta\mathbf{F}_{n+1}(\mathbf{F}_n)^T,$$

$$\mathbf{D}_2 := \mathbf{F}_n^{-T} \delta\mathbf{F}_{n+1},$$

$$\mathbf{D}_3 := (\nabla \otimes \mathbf{w})\mathbf{F}_n^{-2} \delta\mathbf{F}_{n+1},$$

and

$$\mathbf{D}_4 := (\nabla \otimes \mathbf{w})\mathbf{F}_n^{-1},$$

will be implemented as the macros

$$\left. \begin{aligned} &\langle d1Aux11 \rangle, \langle d1Aux12 \rangle, \dots, \langle d1Aux22 \rangle, \\ &\langle d2Aux11 \rangle, \langle d2Aux12 \rangle, \dots, \langle d2Aux22 \rangle \\ &\langle d3Aux11 \rangle, \langle d3Aux12 \rangle, \dots, \langle d3Aux22 \rangle \\ &\langle d4Aux11 \rangle, \langle d4Aux12 \rangle, \dots, \langle d4Aux22 \rangle \end{aligned} \right\}, \quad (9.66)$$

respectively.

In the above notation, the tangent Kirchhoff stress term becomes

$$\frac{\partial \kappa}{\partial \mathbf{F}}(\mathbf{F}_n) \delta \mathbf{F}_{n+1} = \mu \mathbf{D}_1 \quad (9.67)$$

while the weak BVP formulation acquires the form

$$\left. \begin{aligned} 0 &= \int_{\Omega_0} \kappa[\mathbf{F}_n] : \mathbf{D}_4 \\ &- \int_{\Omega_0} \kappa[\mathbf{F}_n] : \mathbf{D}_3 \\ &+ \int_{\Omega_0} \left\{ \frac{\partial \kappa}{\partial \mathbf{F}}[\mathbf{F}_n] \delta \mathbf{F}_{n+1} \right\} : \mathbf{D}_4 \end{aligned} \right\} \quad \text{for all test functions } \mathbf{w} \quad (9.68)$$

9.14 Whispering gallery modes

Written by I. S. Grudinin.

In whispering gallery mode (WGM) resonators, which are typically spheres or disks, electromagnetic field is trapped by total internal reflections from the boundary. Modes of such resonators are distinguished by compact volume and record high quality factors (Q) in a broad range of frequencies. Modern applications of such resonators include microwave and optical cavities for atomic clocks, cavity optomechanics, nonlinear and quantum optics. Analytical solutions for WG modes are only available for a limited number of idealized geometries, such as sphere or ellipsoid. Since resonator dimensions are typically much larger than optical wavelength, direct application of numerical 3D finite difference time domain (FDTD) or finite element methods (FEM) is not practical. It's possible to solve the vectorial wave equation by reducing it to a two dimensional case by taking axial symmetry into account. Such reduction leads to a system of 3 equations to be solved in a 2D “ $\rho - z$ ” section of a resonator. Please refer to [46] for a detailed derivation and to [47] for an example of using `FreeFem++` to compute WGMs.

9.14.1 Wave equation for the WGMs

Since electric field is discontinuous on the surface of a dielectric and magnetic field is typically not, we derive our equations for the magnetic field. The electric field can be easily derived at a later stage from $\mathbf{E} = \frac{i}{\omega \epsilon_0} \hat{\epsilon}^{-1} \nabla \times \mathbf{H}$. Following a standard procedure starting with Maxwell equations we derive a wave equation in a single-axis anisotropic medium such as an optical crystal:

$$\nabla \times (\hat{\epsilon}^{-1} \nabla \times \mathbf{H}) - k_0^2 \mathbf{H} - \alpha \nabla (\nabla \cdot \mathbf{H}) = 0 \quad (9.69)$$

Here $k_0 = \omega/c$ is the wavenumber, α is the penalty term added to fight spurious FEM solutions. For anisotropic single-axis medium with $\partial \hat{\epsilon} / \partial \phi = 0$ in cylindrical system of coordinates we have:

$$\hat{\epsilon} = \begin{pmatrix} \epsilon_\rho & 0 & 0 \\ 0 & \epsilon_\rho & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}.$$

We now assume axial symmetry of our electromagnetic fields and insert an imaginary unity in front of the H_ϕ to allow all field components to be real numbers and also to account for the phase shift of this component $\mathbf{H}(\rho, \phi, z) = \{H_\rho(\rho, z), iH_\phi(\rho, z), H_z(\rho, z)\} \times e^{im\phi}$.

We write the wave equation (9.69) explicitly in cylindrical coordinates, thus obtaining a set of three differential equations for the domain Ω given by the resonator's cross section and some space

outside:

$$\begin{aligned} A_1\{H_\rho^t, H_\phi^t, H_z^t\} &= 0 \\ A_2\{H_\rho^t, H_\phi^t, H_z^t\} &= 0 \\ A_3\{H_\rho^t, H_\phi^t, H_z^t\} &= 0 \end{aligned} \tag{9.70}$$

The numerical solutions of these equations and boundary conditions can be found with `FreeFem++` if we write the system in the weak, or integral form.

9.15 Weak formulation

In general, to obtain the integral or “weak” statements equivalent to system (9.70) and boundary conditions we form a scalar dot product between an arbitrary magnetic field test function $\mathbf{H}^t = \{H_\rho^t, H_\phi^t, H_z^t\}$ and the components of our vectorial equation A_1, A_2, A_3 , and integrate over the resonator’s cross section domain Ω (and its boundary for the boundary conditions):

$$\int_{\Omega} (H_\rho^t A_1 + H_\phi^t A_2 + H_z^t A_3) d\Omega \tag{9.71}$$

We can reduce the order of partial derivatives in this integral by using the Green’s formula for integration by parts. For example:

$$\int_{\Omega} H_z^t \frac{\partial^2 H_z}{\partial \rho^2} d\Omega = - \int_{\Omega} \frac{\partial H_z^t}{\partial \rho} \frac{\partial H_z}{\partial \rho} d\Omega + \oint H_z^t \frac{\partial H_z}{\partial \rho} n_\rho d\Gamma \tag{9.72}$$

Thus converting equations (9.70) we obtain a large expression for the weak form (see [1])

9.15.1 A dielectric sphere example with `FreeFem++`

We now compute the fundamental mode frequency for a fused silica sphere. The sphere is 36 micrometer in diameter, the refractive index is 1.46, the boundary condition is the magnetic wall (which can actually be omitted as it holds automatically). The example can be found in `examples++-eigenvalue/WGM-sphere.edp` in the distribution.

Chapter 10

MPI Parallel version

A first attempt of parallelization of FreeFem++ is made here with **mpi**. An extended interface with MPI has been added to FreeFem++ version 3.5, (see the MPI documentation for the functionality of the language at <http://www.mpi-forum.org/docs/mpi21-report.pdf>).

10.1 MPI keywords

The following keywords and concepts are used:

mpiComm to defined a *communication world*

mpiGroup to defined a group of *processors* in the communication world

mpiRequest to defined a equest to wait for the end of the communication

10.2 MPI constants

mpisize The total number of *processes*,

mpirank the id-number of my current process in $\{0, \dots, mpisize - 1\}$,

mpiUndefined The MPI_Undefined constant,

mpiAnySource The MPI_ANY_SOURCE constant,

mpiCommWorld The MPI_COMM_WORLD constant ,

... and all the keywords of MPI_Op for the *reduce* operator:

mpiMAX, mpiMIN, mpiSUM, mpiPROD, mpiLAND, mpiLOR, mpiLXOR, mpiBAND,
mpiBXOR.

10.3 MPI Constructor

```
int[int] proc1=[1,2,3],proc2=[0,4];  
mpiGroup grp(procs);           // set MPI.Group to proc 1,2,3 in MPI_COMM_WORLD
```

```

mpiGroup grp1(comm,proc1);           // set MPI_Group to proc 1,2,3 in comm
mpiGroup grp2(grp,proc2);           // set MPI_Group to grp union proc1

mpiComm comm=mpiCommWorld;           // set a MPI_Comm to MPI_COMM_WORLD
mpiComm ncomm(mpiCommWorld,grp);     // set the MPI_Comm form grp
                                     // MPI_COMM_WORLD
mpiComm ncomm(comm,color,key);       // MPI_Comm_split(MPI_Comm comm,
                                     // int color, int key, MPI_Comm *ncomm)
mpiComm ncomm(processor(local_comm,local_leader),
               processor(peer_comm,peer_leader),tag);
// build MPI_INTERCOMM_CREATE(local_comm, local_leader, peer_comm,
// remote_leader, tag, &ncomm)
mpiComm ncomm(intercomm,height) ;    // build using
                                     // MPI_Intercomm_merge( intercomm, high, &ncomm)
mpiRequest rq;                      // defined an MPI_Request
mpiRequest[int] arq(10);           // defined an array of 10 MPI_Request

```

10.4 MPI functions

```

mpiSize(comm) ;                     // return the size of comm (int)
mpiRank(comm) ;                     // return the rank in comm (int)

processor(i) // return processor i with no Resquest in MPI_COMM_WORLD
processor(mpiAnySource) // return processor any source
                                     // with no Resquest in MPI_COMM_WORLD
processor(i,comm) // return processor i with no Resquest in comm
processor(comm,i) // return processor i with no Resquest in comm
processor(i,rq,comm) // return processor i with Resquest rq in comm
processor(i,rq) // return processor i with Resquest rq in
                                     // MPI_COMM_WORLD
processorblock(i) // return processor i in MPI_COMM_WORLD
// in block mode for synchronously communication
processorblock(mpiAnySource) // return processor any source
// in MPI_COMM_WORLD in block mode for synchronously communication
processorblock(i,comm) // return processor i in in comm in block mode

mpiBarrier(comm) ; // do a MPI_Barrier on communicator comm,
mpiWait(rq); // wait on of Request,
mpiWaitAll(arq); // wait add of Request array,
mpiWtime() ; // return MPIWtime in second (real),
mpiWtick() ; // return MPIWTick in second (real),

```

where a processor is just a integer rank, pointer to a MPI_comm and pointer to a MPI_Request, and processorblock with a special MPI_Request.

10.5 MPI communicator operator

```

int status; // to get the MPI status of send / recv
processor(10) << a << b; // send a,b asynchronously to the process 1,
processor(10) >> a >> b; // receive a,b synchronously from the process 10,
broadcast(processor(10,comm),a); // broadcast from processor

```

```

//      of com to other comm processor
status=Send( processor(10,comm) , a); //      send synchronously
Recv}

//      to the process 10 the data a
status=Recv( processor(10,comm) , a); //      receive synchronously
//      from the process 10 the data a;
status=Isend( processor(10,comm) , a); //      send asynchronously to
//      the process 10 , the data a without request
status=Isend( processor(10,rq,comm) , a) ; //      send asynchronously to to
//      the process 10, the data a with request
status=Irecv( processor(10,rq) , a) ; //      receive synchronously from
//      the process 10, the data a;
status=Irecv( processor(10) , a) ; //      Error
//      Error asynchronously without request .
broadcast (processor (comm,a)); //      Broadcast to all process of comm

```

where the data type of a can be of type of int,real, complex, int[int], double[int], complex[int], int[int,int], double[int,int], complex[int,int], mesh, mesh3, mesh[int], mesh3[int], matrix, matrix<complex>

```

processor(10,rq) << a ; //      send asynchronously to the process 10
//      the data a with request
processor(10,rq) >> a ; //      receive asynchronously from the process 10
//      the data a with request

```

If a, b are arrays or full matrices of int, real, or complex, we can use the following MPI functions:

```

mpiAlltoall (a,b[,comm]) ;
mpiAllgather (a,b[,comm]) ;
mpiGather (a,b,processor(..) ) ;
mpiScatter (a,b,processor(..) ) ;
mpiReduce (a,b,processor(..),mpiMAX) ;
mpiAllReduce (a,b,comm, mpiMAX) ;
mpiReduceScatter (a,b,comm, mpiMAX) ;

```

See the examples++-mpi/essai.edp to test of all this functionality and Thank, to Guy-Antoine Atenekeng Kahou, for his help to code this interface.

10.6 Schwarz example in parallel

This example is a rewritting of example schwarz-overlap in section 9.8.1.

```

[examples++-mpi] Hecht%lamboot
LAM 6.5.9/MPI 2 C++/ROMIO - Indiana University
[examples++-mpi] hecht% mpirun -np 2 FreeFem++-mpi schwarz-c.edp

```

```

//      a new coding version c, methode de schwarz in parallele
//      with 2 proc.
//      -----
//      F.Hecht december 2003
//      -----
//      to test the broadcast instruction
//      and array of mesh

```

```

//      add add the stop test
//      -----

if ( mpisize != 2 ) {
    cout << " sorry, number of processors !=2 " << endl;
    exit(1);}
verbosity=3;
int interior = 2;
int exterior = 1;
border a(t=1,2){x=t;y=0;label=exterior;};
border b(t=0,1){x=2;y=t;label=exterior;};
border c(t=2,0){x=t ;y=1;label=exterior;};
border d(t=1,0){x = 1-t; y = t;label=interior;};
border e(t=0, pi/2){ x= cos(t); y = sin(t);label=interior;};
border e1(t=pi/2, 2*pi){ x= cos(t); y = sin(t);label=exterior;};
int n=4;
mesh[int] Th(mpisize);
if (mpirank == 0)
    Th[0] = buildmesh( a(5*n) + b(5*n) + c(10*n) + d(5*n));
else
    Th[1] = buildmesh ( e(5*n) + e1(25*n) );

broadcast(processor(0),Th[0]);
broadcast(processor(1),Th[1]);

fespace Vh(Th[mpirank],P1);
fespace Vhother(Th[1-mpirank],P1);

Vh u=0,v;
Vhother U=0;
int i=0;

problem pb(u,v,init=i,solver=Cholesky) =
    int2d(Th[mpirank])( dx(u)*dx(v)+dy(u)*dy(v) )
    - int2d(Th[mpirank])( v)
    + on(interior,u = U)  + on(exterior,u= 0 ) ;

for ( i=0 ;i< 20; i++)
{
    cout << mpirank << " loop " << i << endl;
    pb;

    //      send u to the other proc, receive in U
    processor(1-mpirank) << u[];  processor(1-mpirank) >> U[];
    real err0,err1;
    err0 = int1d(Th[mpirank],interior)(square(U-u)) ;
    //      send err0 to the other proc, receive in err1
    processor(1-mpirank)<<err0;  processor(1-mpirank)>>err1;
    real err= sqrt(err0+err1);
    cout <<" err = " << err << " err0 = " << err0
        << ", err1 = " << err1 << endl;
    if(err<1e-3) break;
};
if (mpirank==0)
    plot(u,U,ps="uU.eps");

```

10.6.1 True parallel Schwarz example

This is a explanation of the two script `examples++-mpi/MPIGMRES[2]D.edp`, a Schwarz parallel with a complexity almost independent of the number of process (with a coarse grid preconditioner).

Thank to F. Nataf.

To solve the following Poisson problem on domain Ω with boundary Γ in $L^2(\Omega)$:

$$-\Delta u = f, \text{ in } \Omega, \text{ and } u = g \text{ on } \Gamma,$$

where f and g are two given functions of $L^2(\Omega)$ and of $H^{\frac{1}{2}}(\Gamma)$,

Let introduce $(\pi_i)_{i=1,\dots,N_p}$ a regular partition of the unity of Ω , q-e-d:

$$\pi_i \in \mathcal{C}^0(\Omega) : \quad \pi_i \geq 0 \text{ and } \sum_{i=1}^{N_p} \pi_i = 1.$$

Denote Ω_i the sub domain which is the support of π_i function and also denote Γ_i the boundary of Ω_i .

The parallel Schwarz method is Let $\ell = 0$ the iterator and a initial guest u^0 respecting the boundary condition (i.e. $u|_{\Gamma}^0 = g$).

$$\forall i = 1.., N_p : \quad -\Delta u_i^\ell = f, \text{ in } \Omega_i, \quad \text{and } u_i^\ell = u^\ell \text{ on } \Gamma_i \setminus \Gamma, \quad u_i^\ell = g \text{ on } \Gamma_i \cap \Gamma \quad (10.1)$$

$$u^{\ell+1} = \sum_{i=1}^{N_p} \pi_i u_i^\ell \quad (10.2)$$

After discretization with the Lagrange finite element method, with a compatible mesh \mathcal{T}_{hi} of Ω_i , i. e., the exist a global mesh \mathcal{T}_h such that \mathcal{T}_{hi} is include in \mathcal{T}_h . Let us denote:

- V_{hi} the finite element space corresponding to domain Ω_i ,
- \mathcal{N}_{hi} is the set of the degree of freedom σ_i^k ,
- $\mathcal{N}_{hi}^{\Gamma_i}$ is the set of the degree of freedom of V_{hi} on the boundary Γ_i of Ω_i ,
- $\sigma_i^k(v_h)$ is the value the degree of freedom k ,
- $V_{0hi} = \{v_h \in V_{hi} : \forall k \in \mathcal{N}_{hi}^{\Gamma_i}, \quad \sigma_i^k(v_h) = 0\}$,
- the conditional expression $a ? b : c$ is defined like in C of C++ language by

$$a ? b : c \equiv \begin{cases} \text{if } a \text{ is true then return } b \\ \text{else return } c \end{cases}.$$

Remark we never use finite element space associated to the full domain Ω because it to expensive.

We have to defined to operator to build the previous algorithm:

We denote $u_{h|i}^\ell$ the restriction of u_h^ℓ on V_{hi} , so the discrete problem on Ω_i of problem (10.1) is find $u_{h|i}^\ell \in V_{hi}$ such that: where g_i^k is the value of g associated to the degree of freedom $k \in \mathcal{N}_{hi}^{\Gamma_i}$.

In FreeFem++ , it can be written has with U is the vector corresponding to $u_{h|i}^\ell$ and the vector $U1$ is the vector corresponding to $u_{h|i}^\ell$ is the solution of:

```
real[int] U1(Ui.n);
```

```

real[int] b= onG .* U;
b  = onG ? b : Bi ;
U1 = Ai-1*b;

```

where $\text{onG}[i] = (i \in \Gamma_i \setminus \Gamma) ? 1 : 0$, and Bi the right of side of the problem, are defined by

```

fespace Whi(Thi,P2); // def of the Finite element space.
varf vPb(U,V)= int3d(Thi)(grad(U)'*grad(V)) + int3d(Thi)(F*V) +on(1,U=g) + on(10,U=G);
varf vPbon(U,V)=on(10,U=1)+on(1,U=0);
matrix Ai = vPb(Whi,Whi,solver=sparsesolver);
real[int] onG = vPbon(0,Whi);
real[int] Bi=vPb(0,Whi);

```

where the FreeFem++ label of Γ is 1 and the label of $\Gamma_i \setminus \Gamma$ is 10.

To build the transfer/update part corresponding to (10.2) equation on process i , let us call njpart the number the neighborhood of domain of Ω_i (i.e: π_j is none 0 of Ω_i), we store in an array jpart of size njpart all this neighborhood. Let us introduce two array of matrix, $\text{Smj}[j]$ to defined the vector to send from i to j a neighborhood process, and the matrix $\text{rMj}[j]$ to after to reduce owith neighborhood j domain.

So the tranfert and update part compute $v_i = \pi_i u_i + \sum_{j \in J_i} \pi_j u_j$ and can be write the FreeFem++ function Update:

```

func bool Update(real[int] &ui, real[int] &vi)
{ int n= jpart.n;
  for(int j=0;j<njpart;++j) Usend[j][]=sMj[j]*ui;
  mpiRequest[int] rq(n*2);
  for (int j=0;j<n;++j) Irecv(processor(jpart[j],comm,rq[j ]), Ri[j][]);
  for (int j=0;j<n;++j) Isend(processor(jpart[j],comm,rq[j+n]), Si[j][]);
  for (int j=0;j<n*2;++j) int k= mpiWaitAny(rq);
  // apply the unity local partition .
  vi = Pii*ui; // set to  $\pi_i u_i$ 
  for(int j=0;j<njpart;++j) vi += rMj[j]*Vrecv[j][]; // add  $\pi_j u_j$ 
  return true; }

```

where the buffer are defined by:

```

InitU(njpart,Whij,Thij,aThij,Usend) // defined the send buffer
InitU(njpart,Whij,Thij,aThij,Vrecv) // defined the revc buffer

```

with the following macro definition:

```

macro InitU(n,Vh,Th,aTh,U) Vh[int] U(n); for(int j=0;j<n;++j) {Th=aTh[j]; U[j]=0;}
//

```

First gmres algorithm: you can easily accelerate the fixe point algorithm by using a parallel GMRES algorithm after the introduction the following affine \mathcal{A}_i operator sub domain Ω_i .

```

func real[int] DJ0(real[int] &U) {
  real[int] V(U.n) , b= onG .* U;
  b  = onG ? b : Bi ;
  V = Ai-1*b;
  Update(V,U);
}

```



```
V -= U;    return V; }
```

Where the parallel MPMRES or MPICG algorithm is just a simple way to solve in parallel the following $A_i x_i = b_i, i = 1, \dots, N_p$ by just changing the dot product by reduce the local dot product of all process with the following MPI code:

```
template<class R> R ReduceSum1(R s,MPI_Comm * comm)
{
    R r=0;
    MPI_Allreduce( &s, &r, 1 ,MPI_TYPE<R>::TYPE(),    MPI_SUM,    *comm );
    return r; }
```

This is done in MPIGC dynamics library tool.

Second gmres algorithm: Use schwarz algorithm as a preconditioner of basic GMRES method to solving the parallel problem.

```
func real[int] DJ(real[int]& U)                                // the original problem
{
    ++kiter;
    real[int] V(U.n);
    V = Ai*U;
    V = onGi ? 0.: V;                                         // remove boundary term ...
    return V;
}

func real[int] PDJ(real[int]& U)                                // the preconditioner
{
    real[int] V(U.n);
    real[int] b= onG ? 0. : U;
    V = Ai^-1*b;
    Update(V,U);
    return U;
}
```

Third gmres algorithm: Add a coarse solver to the previous algorithm
First build a coarse grid on processor 0, and the

```
matrix AC,Rci,Pci;                                           //
if(mpiRank(comm)==0)
    AC = vPbC(VhC,VhC,solver=sparsesolver);                // the corase problem

Pci= interpolate(Whi,VhC);                                     // the projection on coarse grid.
Rci = Pci'*Pii; // the Restriction on Process i grid with the partition  $\pi_i$ 

func bool CoarseSolve(real[int]& V,real[int]& U,mpiComm& comm)
{
    // solvibg the coarse probleme

    real[int] Uc(Rci.n),Bc(Uc.n);
    Uc= Rci*U;
    mpiReduce(Uc,Bc,processor(0,comm),mpiSUM);
    if(mpiRank(comm)==0)
        Uc = AC^-1*Bc;
    broadcast(processor(0,comm),Uc);
    V = Pci*Uc;
}
```

The New preconditionner

```

func real[int] PDJC(real[int]& U)                                     //
{ //      Precon C1= Precon //, C2 precon Coarse
//      Idea : F. Nataf.
//      0 ~ (I C1A) (I-C2A) => I ~ - C1AC2A +C1A +C2A
//      New Prec P= C1+C2 - C1AC2 = C1(I- A C2) +C2
//      ( C1(I- A C2) +C2 ) Uo
//      V = - C2*Uo
//      ....
real[int] V(U.n);
CoarseSolve(V,U,comm);
V = -V;                                                         //      -C2*Uo
U += Ai*V;                                                         //      U = (I-A C2) Uo
real[int] b= onG ? 0. : U;
U = Ai^-1*b;                                                         //      ( C1( I -A C2) Uo
V = U -V;                                                         //
Update(V,U);
return U;
}

```

The code to the 4 algorithms:

```

real epss=1e-6;
int rgmres=0;
if(gmres==1)
{
    rgmres=MPIAffineGMRES(DJ0,u[],veps=epss,nbiter=300,comm=comm,
                          dimKrylov=100,verbosity=ipart ? 0: 50);
    real[int] b= onG .* u[];
    b = onG ? b : Bi ;
    v[] = Ai^-1*b;
    Update(v[],u[]);
}
else if (gmres==2)
    rgmres= MPILinearGMRES(DJ,precon=PDJ,u[],Bi,veps=epss,nbiter=300,comm=comm
                          ,dimKrylov=100,verbosity=ipart ? 0: 50);
else if (gmres==3)
    rgmres= MPILinearGMRES(DJ,precon=PDJC,u[],Bi,veps=epss,nbiter=300,comm=comm,
                          dimKrylov=100,verbosity=ipart ? 0: 50);
else
    //      algo Shwarz for demo ...
    for(int iter=0;iter <10; ++iter)
        ....

```

We have all ingredient to solve in parallel if we have et the partitions of the unity. To build this partition we do: the initial step on process 1 tp build a coarse mesh, \mathcal{T}_h^* of the full domain, and build the partition π function constant equal to i on each sub domain $\mathcal{O}_i, i = 1, \dots, N_p$, of the grid with the Metis graph partitioner [?] and on each process i in $1.., N_p$ do

1. Broadcast from process 1, the mesh \mathcal{T}_h^* (call Thii in FreeFem++ script), and π function,
2. remark that the characteristic function $\mathbb{1}_{\mathcal{O}_i}$ of domain \mathcal{O}_i , is defined by $(\pi = i)?1 : 0$,

3. let us call Π_P^2 (resp. Π_V^2) the L^2 on P_h^* the space of the constant finite element function per element on \mathcal{T}_h^* (resp. V_h^* the space of the affine continuous finite element per element on \mathcal{T}_h^*). and build in parallel the π_i and Ω_i , such that $\mathcal{O}_i \subset \Omega_i$ where $\mathcal{O}_i = \text{supp}((\Pi_V^2 \Pi_C^2)^m \mathbf{1}_{\mathcal{O}_i})$, and m is the overlaps size on the coarse mesh (generally one),

(this is done in function `AddLayers(Thii, suppii[], nlayer, phii[])`; We choose a function $\pi_i^* = (\Pi_1^2 \Pi_0^2)^m \mathbf{1}_{\mathcal{O}_i}$ so the partition of the unity is simply defined by

$$\pi_i = \frac{\pi_i^*}{\sum_{j=1}^{N_p} \pi_j^*} \quad (10.3)$$

The set J_i of neighborhood of the domain Ω_i , and the local version on V_{hi} can be defined the array `jpart` and `njpart` with:

```
Vhi pii=pi_i* ;   Vhi[int] pij(npij);   //   local partition of 1 = pii + \sum_j
pij[j]

int[int] jpart(npart);   int njpart=0;
Vhi sumphi = pi_i* ;
for (int i=0; i<npart; ++i)
  if(i != ipart ) {
    if(int3d(Thi)( pi_j^*)>0) {
      pij[njpart]=pi_j^*;
      sumphi[] += pij[njpart][];
      jpart[njpart++]=i;}}}
pii[]=pii[] ./ sumphi[];
for (int j=0; j<njpart; ++j) pij[j][] = pij[j][] ./ sumphi[];
jpart.resize(njpart);
```

4. We call $\mathcal{T}_{h_{ij}}^*$ the sub mesh part of \mathcal{T}_{hi} where π_j are none zero. and tank to the function `trunc` to build this array,

```
for(int jp=0; jp<njpart; ++jp)
  aThij[jp] = trunc(Thi, pij[jp]>1e-10, label=10);
```

5. At this step we have all on the coarse mesh , so we can build the fine final mesh by splitting all meshes : `Thi`, `Thij[j]`, `Thij[j]` with `FreeFem++ trunc` mesh function which do restriction and slipping.

6. The construction of the send/recv matrices `sMj` and `rMj` : can done with this code:

```
mesh3 Thij=Thi; //   variable meshes
fespace Whij(Thij, Pk); //   variable fespace ..
matrix Pii; Whi wpii=pii; Pii = wpii[]; //   Diagonal matrix
corresponding \times \pi_i
matrix[int] sMj(njpart), rMj(njpart); //   M send/rend case..
for(int jp=0; jp<njpart; ++jp)
{ int j=jpart[jp];
  Thij = aThij[jp]; //   change mesh to change Whij, Whij
  matrix I = interpolate(Whij, Whi); //   Whij <- Whi
  sMj[jp] = I*Pii; //   Whi -> s Whij
  rMj[jp] = interpolate(Whij, Whi, t=1); //   Whij -> Whi
}
```

To build a not too bad application, I have added code to change variable from parameter value with the following code

```
include "getARGV.idp"
verbosity=getARGV("-vv",0);
int vdebug=getARGV("-d",1);
int ksplit=getARGV("-k",10);
int nloc = getARGV("-n",25);
string sff=getARGV("-p","");
int gmres=getARGV("-gmres",3);
bool dplot=getARGV("-dp",0);
int nC = getARGV("-N",max(nloc/10,4));
```

And small include to make graphic in parallel of distributed solution of vector u on mesh T_h with the following interface:

```
include "MPIplot.idp"
func bool plotMPIall(mesh &Th,real[int] & u,string cm)
{ PLOTMPIALL(mesh,Pk, Th, u,{ cmm=cm,nbiso=20,fill=1,dim=3,value=1}); return 1; }
```

remark the {cmm=cm, ... =1} in the macro argument is a way to quote macro argument so the argument is cmm=cm, ... =1.

Chapter 11

Parallel sparse solvers

Parallel sparse solvers use several processors to solve linear systems of equation. Like sequential, parallel linear solvers can be direct or iterative. In `FreeFem++` both are available.

11.1 Using parallel sparse solvers in **FreeFem++**

We recall that the `solver` parameters are defined in the following commands: `solve`, `problem`, `set` (setting parameter of a matrix) and in the construction of the matrix corresponding to a bilinear form. In these commands, the parameter `solver` must be set to `sparsesolver` for parallel sparse solver. We have added specify parameters to these command lines for parallel sparse solvers. These are

- `lparams`: vector of integer parameters (l is for the c++ type long)
- `dparams`: vector of real parameters
- `sparams`: string parameters
- `datafilename`: name of the file which contains solver parameters

The following four parameters are only for direct solvers and are vectors. These parameters allow the user to preprocess the matrix (see the section on sparse direct solver above for more information).

- `permr`: row permutation (integer vector)
- `permc`: column permutation or inverse row permutation (integer vector)
- `scaler`: row scaling (real vector)
- `scalec`: column scaling (real vector)

There are two possibilities to control solver parameters. The first method defines parameters with `lparams`, `dparams` and `sparams` in `.edp` file. The second one reads the solver parameters from a data file. The name of this file is specified by `datafilename`. If `lparams`, `dparams`, `sparams` or `datafilename` is not provided by the user, the solver's default value is used.

To use parallel solver in `FreeFem++`, we need to load the dynamic library corresponding to this solver. For example to use MUMPS solver as parallel solver in `FreeFem`, write in the `.edp` file **`load "MUMPS_FreeFem"`**.

If the libraries are not loaded, the default sparse solver will be loaded (default sparse solver is UMFPACK). The table 11.1 gives this new value for the different libraries.

Libraries	default sparse solver	
	real	complex
MUMPS_FreeFem	mumps	mumps
real_SuperLU_DIST_FreeFem	SuperLU_DIST	previous solver
complex_SuperLU_DIST_FreeFem	previous solver	SuperLU_DIST
real_pastix_FreeFem	pastix	previous solver
complex_pastix_FreeFem	previous solver	pastix
hips_FreeFem	hips	previous solver
hypre_FreeFem	hypre	previous solver
parms_FreeFem	parms	previous solver

Table 11.1: Default sparse solver for real and complex arithmetics when we load a parallel sparse solver library

We also add functions (see Table 11.2) with no parameter to change the default sparse solver in the .edp file. To use these functions, we need to load the library corresponding to the solver. An example of using different parallel sparse solvers for the same problem is given in testdirectsolvers.edp (directory example+ + -mpi).

function	default sparse solver	
	real	complex
defaulttoMUMPS()	mumps	mumps
realdefaulttoSuperLUDist()	SuperLU_DIST	previous solver
complexdefaulttoSuperLUDist()	previous solver	SuperLU_DIST
realdefaulttopastix()	pastix	previous solver
complexdefaulttopastix()	previous solver	pastix
defaulttohips()	hips	previous solver
defaulttohypre()	hypre	previous solver
defaulttoparms()	parms	previous solver

Table 11.2: Functions that allow to change the default sparse solver for real and complex arithmetics and the result of these functions

Example 11.1 (testdirectsolvers.edp)

```

load "../src/solver/MUMPS_FreeFem"
// default solver : real-> MUMPS, complex -> MUMPS
load "../src/solver/real_SuperLU_DIST_FreeFem"
// default solver : real-> SuperLU_DIST, complex -> MUMPS
load "../src/solver/real_pastix_FreeFem"
// default solver : real-> pastix, complex -> MUMPS

// solving with pastix
{
  matrix A =
    [[ 1, 2, 2, 1, 1],
     [ 2, 12, 0, 10, 10],
     [ 2, 0, 1, 0, 2],
     [ 1, 10, 0, 22, 0],
     [ 1, 10, 2, 0, 22]];

```

```

real[int] xx = [ 1,32,45,7,2], x(5), b(5), di(5);
b=A*xx;
cout << "b=" << b << endl;
cout << "xx=" << xx << endl;

set(A,solver=sparsesolver,datafilename="ffpastix_iparm_dparm.txt");
cout << "solving solution" << endl;
x = A^-1*b;
cout << "b=" << b << endl;
cout << "x=" << endl; cout << x << endl;
di = xx-x;
if(mpirank==0){
cout << "x-xx="<< endl; cout << "Linf "<< di.linf << " L2 " << di.l2 << endl;
}
}

// solving with SuperLU_DIST
realdefaulttoSuperLUdist();
// default solver : real-> SuperLU_DIST, complex -> MUMPS
{
matrix A =
[[ 1, 2, 2, 1, 1],
[ 2, 12, 0, 10, 10],
[ 2, 0, 1, 0, 2],
[ 1, 10, 0, 22, 0.],
[ 1, 10, 2, 0., 22]];

real[int] xx = [ 1,32,45,7,2], x(5), b(5), di(5);
b=A*xx;
cout << "b=" << b << endl;
cout << "xx=" << xx << endl;

set(A,solver=sparsesolver,datafilename="ffsuperlu_dist_fileparam.txt");
cout << "solving solution" << endl;
x = A^-1*b;
cout << "b=" << b << endl;
cout << "x=" << endl; cout << x << endl;
di = xx-x;
if(mpirank==0){
cout << "x-xx="<< endl; cout << "Linf "<< di.linf << " L2 " << di.l2 << endl;
}
}

// solving with MUMPS
defaulttoMUMPS();
// default solver : real-> MUMPS, complex -> MUMPS
{
matrix A =
[[ 1, 2, 2, 1, 1],
[ 2, 12, 0, 10, 10],
[ 2, 0, 1, 0, 2],
[ 1, 10, 0, 22, 0.],
[ 1, 10, 2, 0., 22]];

real[int] xx = [ 1,32,45,7,2], x(5), b(5), di(5);

```

```

b=A*xx;
cout << "b=" << b << endl;
cout << "xx=" << xx << endl;

set(A,solver=sparseSolver,datafilename="ffmumps_fileparam.txt");
cout << "solving solution" << endl;
x = A^-1*b;
cout << "b=" << b << endl;
cout << "x=" << endl; cout << x << endl;
di = xx-x;
if(mpirank==0){
cout << "x-xx="<< endl; cout << "Linf "<< di.linf << " L2 " << di.l2 << endl;
}
}

```

11.2 Sparse direct solver

In this section, we present the sparse direct solvers interfaced with FreeFem++ .

11.2.1 MUMPS solver

Multifrontal Massively Parallel Solver (MUMPS) is a free library [?, ?, ?]. This package solves linear system of the form $Ax = b$ where A is a square sparse matrix with a direct method. The square matrix considered in MUMPS can be either unsymmetric, symmetric positive definite or general symmetric. The method implemented in MUMPS is a direct method based on a multifrontal approach [?]. It constructs a direct factorization $A = LU$, $A = L^t DL$ depending of the symmetry of the matrix A . MUMPS uses the following libraries : BLAS[?, ?], BLACS and ScaLAPACK[?].

Remark 7 *MUMPS does not solve linear system with a rectangular matrix.*

Installation of MUMPS To used MUMPS in FreeFem++ , you have to install the MUMPS package into your computer. MUMPS is written in Fortran 90. The parallel version is constructed using MPI [?] for message passing and BLAS [?, ?], BLACS and ScaLAPACK[?]. Therefore, a fortran compiler is needed, and MPI, BLAS, BLACS and ScaLAPACK . An installation procedure to obtain this package is given in the file README_COMPILE in the directory src/solver of FreeFem++ .

Creating Library of MUMPS interface for FreeFem++ : The MUMPS interface for FreeFem++ is given in file MUMPS_freeFem.cpp (directory src/solver/). This interface works with the release 3.8.3 and 3.8.4 of MUMPS. To used MUMPS in FreeFem++ , we need the library corresponding to this interface. A description to obtain this library is given in the file README_COMPILE in the directory src/solver of FreeFem++ . We recall here the procedure. Go to the directory src/solver in FreeFem++ package. Edit the file makefile-sparseSolver.inc to yours system: comment Section 1, comment line corresponding to libraries BLAS, BLACS, ScaLAPACK, Metis, scotch in Section 2 and comment in Section 3 the paragraph corresponding to MUMPS solver. And then type **make mumps** in a terminal window.

Now we give a short description of MUMPS parameters before describing the method to call MUMPS in FreeFem++ .

MUMPS parameters: There are four input parameters in MUMPS (see [?]). Two integers SYM and PAR, a vector of integer of size 40 INCTL and a vector of real of size 15 CNTL. The first parameter gives the type of the matrix: 0 for unsymmetric matrix, 1 for symmetric positive matrix and 2 for general symmetric. The second parameter defined if the host processor work during the factorization and solves steps : PAR=1 host processor working and PAR=0 host processor not working. The parameter INCTL and CNTL is the control parameter of MUMPS. The vectors ICNTL and CNTL in MUMPS becomes with index 1 like vector in fortran. A short description of all parameters of ICNTL and CNTL is given in `ffmumps_fileparam.txt`. For more details see the users' guide [?].

We describe now some elements of the main parameters of ICNTL for MUMPS.

Input matrix parameter The input matrix is controlled by parameters ICNTL(5) and ICNTL(18). The matrix format (resp. matrix pattern and matrix entries) are controlled by ICNTL(5) (resp. ICNTL(18)). The different values of ICNTL(5) are 0 for assembled format and 1 for element format. In the current release of FreeFem++, we consider that FE matrix or matrix is storage in assembled format. Therefore, ICNTL(5) is treated as 0 value. The main option for ICNTL(18): ICNTL(18)=0 centrally on the host processor, ICNTL(18)=3 distributed the input matrix pattern and the entries (recommended option for distributed matrix by developer of MUMPS). For other values of ICNTL(18) see the user's guide of MUMPS. These values can be used also in FreeFem++ .

The default option implemented in FreeFem++ are ICNTL(5)=0 and ICNTL(18)=0.

Preprocessing parameter The preprocessed matrix A_p that will be effectively factored is defined by

$$A_p = P D_r A Q_c D_c P^t$$

where P is the permutation matrix, Q_c is the column permutation, D_r and D_c are diagonal matrix for respectively row and column scaling. The ordering strategy to obtain P is controlled by parameter ICNTL(7). The permutation of zero free diagonal Q_c is controlled by parameter ICNTL(6). The row and column scaling is controlled by parameter ICNTL(18). These option are connected and also strongly related with ICNTL(12) (see documentation of mumps for more details [?]). The parameters `permr`, `scaler`, and `scalec` in FreeFem++ allow to give permutation matrix(P), row scaling (D_r) and column scaling (D_c) of the user respectively.

Calling MUMPS in FreeFem++ To call MUMPS in FreeFem++ , we need to load the dynamic library `MUMPS_freefem.dylib` (MacOSX), `MUMPS_freefem.so` (Unix) or `MUMPS_freefem.dll` (Windows). This is done in typing `load "MUMPS_freefem"` in the `.edp` file. We give now the two methods to give the option of MUMPS solver in FreeFem++ .

Solver parameters is defined in .edp file: In this method, we need to give the parameters `lparams` and `dparams`. These parameters are defined for MUMPS by

$$\begin{aligned} & \text{lparams}[0] = \text{SYM}, \\ & \text{lparams}[1] = \text{PAR}, \\ \forall i = 1, \dots, 40, & \quad \text{lparams}[i+1] = \text{ICNTL}(i). \\ \forall i = 1, \dots, 15, & \quad \text{dparams}[i-1] = \text{CNTL}(i). \end{aligned}$$

Reading solver parameters on a file: The structure of data file for MUMPS in FreeFem++ is : first line parameter SYM and second line parameter PAR and in the following line the different value of vectors ICNTL and CNTL. An example of this parameter file is given in `ffmumpsfileparam.txt`.

```

0          /* SYM :: 0 for non symmetric matrix, 1 for symmetric definite positive
matrix and 2 general symmetric matrix*/
1          /* PAR :: 0 host not working during factorization and solves steps, 1
host working during factorization and solves steps*/
-1         /* ICNTL(1) :: output stream for error message */
-1         /* ICNTL(2) :: output for diagnostic printing, statics and warning message
*/
-1         /* ICNTL(3) :: for global information */
0          /* ICNTL(4) :: Level of printing for error, warning and diagnostic message
*/
0          /* ICNTL(5) :: matrix format : 0 assembled format, 1 elemental format.
*/
7          /* ICNTL(6) :: control option for permuting and/or scaling the matrix
in analysis phase */
3          /* ICNTL(7) :: pivot order strategy : AMD, AMF, metis, pord scotch*/
77         /* ICNTL(8) :: Row and Column scaling strategy */
1          /* ICNTL(9) :: 0 solve Ax = b, 1 solve the transposed system A^t x =
b : parameter is not considered in the current release of freefem++*/
0          /* ICNTL(10) :: number of steps of iterative refinement */
0          /* ICNTL(11) :: statics related to linear system depending on ICNTL(9)
*/
1          /* ICNTL(12) :: constrained ordering strategy for general symmetric matrix
*/
0          /* ICNTL(13) :: method to control splitting of the root frontal matrix
*/
20         /* ICNTL(14) :: percentage increase in the estimated working space (default
20%) */
0          /* ICNTL(15) :: not used in this release of MUMPS */
0          /* ICNTL(16) :: not used in this release of MUMPS */
0          /* ICNTL(17) :: not used in this release of MUMPS */
3          /* ICNTL(18) :: method for given : matrix pattern and matrix entries
: */
0          /* ICNTL(19) :: method to return the Schur complement matrix */
0          /* ICNTL(20) :: right hand side form ( 0 dense form, 1 sparse form) :
parameter will be set to 0 for freefem++ */
0          /* ICNTL(21) :: 0, 1 kept distributed solution : parameter is not considered
in the current release of freefem++ */
0          /* ICNTL(22) :: controls the in-core/out-of-core (OOC) facility */
0          /* ICNTL(23) :: maximum size of the working memory in Megabyte than MUMPS
can allocate per working processor */
0          /* ICNTL(24) :: control the detection of null pivot */
0          /* ICNTL(25) :: control the computation of a null space basis */
0          /* ICNTL(26) :: This parameter is only significant with Schur option
(ICNTL(19) not zero). : parameter is not considered in the current release of freefem++
*/
-8         /* ICNTL(27) (Experimental parameter subject to change in next release
of MUMPS) :: control the blocking factor for multiple righthand side during the
solution phase : parameter is not considered in the current release of freefem++
*/

```

```

0      /* ICNTL(28) :: not used in this release of MUMPS*/
0      /* ICNTL(29) :: not used in this release of MUMPS*/
0      /* ICNTL(30) :: not used in this release of MUMPS*/
0      /* ICNTL(31) :: not used in this release of MUMPS*/
0      /* ICNTL(32) :: not used in this release of MUMPS*/
0      /* ICNTL(33) :: not used in this release of MUMPS*/
0      /* ICNTL(34) :: not used in this release of MUMPS*/
0      /* ICNTL(35) :: not used in this release of MUMPS*/
0      /* ICNTL(36) :: not used in this release of MUMPS*/
0      /* ICNTL(37) :: not used in this release of MUMPS*/
0      /* ICNTL(38) :: not used in this release of MUMPS*/
1      /* ICNTL(39) :: not used in this release of MUMPS*/
0      /* ICNTL(40) :: not used in this release of MUMPS*/
0.01   /* CNTL(1) :: relative threshold for numerical pivoting */
1e-8   /* CNTL(2) :: stopping criteria for iterative refinement */
-1     /* CNTL(3) :: threshold for null pivot detection */
-1     /* CNTL(4) :: determine the threshold for partial pivoting */
0.0    /* CNTL(5) :: fixation for null pivots */
0      /* CNTL(6) :: not used in this release of MUMPS */
0      /* CNTL(7) :: not used in this release of MUMPS */
0      /* CNTL(8) :: not used in this release of MUMPS */
0      /* CNTL(9) :: not used in this release of MUMPS */
0      /* CNTL(10) :: not used in this release of MUMPS */
0      /* CNTL(11) :: not used in this release of MUMPS */
0      /* CNTL(12) :: not used in this release of MUMPS */
0      /* CNTL(13) :: not used in this release of MUMPS */
0      /* CNTL(14) :: not used in this release of MUMPS */
0      /* CNTL(15) :: not used in this release of MUMPS */

```

If no solver parameter is given, we used default option of MUMPS solver.

example A simple example of calling MUMPS in FreeFem++ with this two methods is given in the file `testsolver_MUMPS.edp` in the directory `examples++-mpi`.

11.2.2 SuperLU distributed solver

The package `SuperLU_DIST` [?, ?] solves linear systems using LU factorization. It is a free scientific library under BSD license. The web site of this project is <http://crd.lbl.gov/~xiaoye/SuperLU>. This library provides functions to handle square or rectangular matrix in real and complex arithmetics. The method implemented in `SuperLU_DIST` is a supernodal method [?]. New release of this package includes a parallel symbolic factorization [?]. This scientific library is written in C and MPI for communications.

Installation of SuperLU_DIST: To use `SuperLU_DIST` in FreeFem++ , you have to install `SuperLU_DIST` package. We need MPI and ParMetis library to do this compilation. An installation procedure to obtain this package is given in the file `README_COMPILE` in the directory `src/solver/` of the FreeFem++ package.

Creating Library of SuperLU_DIST interface for FreeFem++ : The FreeFem++ interface to SuperLU_DIST for real (resp. complex) arithmetics is given in file `real_SuperLU_DIST_FreeFem.cpp` (resp. `complex_SuperLU_DIST_FreeFem.cpp`). These files are in the directory `src/solver/`. These interfaces are compatible with the release 3.2.1 of SuperLU_DIST. To use SuperLU_DIST in FreeFem++ , we need libraries corresponding to these interfaces. A description to obtain these libraries is given in the file `README_COMPILE` in the directory `src/solver` of FreeFem++ . We recall here the procedure. Go to the directory `src/solver` in FreeFem++ package. Edit the file `makefile-sparsesolver.inc` in your system : comment Section 1, comment line corresponding to libraries BLAS, Metis, ParMetis in Section 2 and comment in Section 3 the paragraph corresponding to SuperLU_DIST solver. And just type **make rsludist** (resp. **make csludist**) in the terminal to obtain the dynamic library of interface for real (resp. complex) arithmetics.

Now we give a short description of SuperLU_DIST parameters before describing the method to call SuperLU_DIST in FreeFem++ .

SuperLU_DIST parameters: We describe now some parameters of SuperLU_DIST. The SuperLU_DIST library use a 2D-logical process group. This process grid is specifies by *nprow* (process row) and *npcol* (process column) such that $N_p = nprow \cdot npcol$ where N_p is the number of all process allocated for SuperLU_DIST.

The input matrix parameters is controlled by "matrix=" in `sparams` for internal parameter or in the third line of parameters file. The different value are

matrix = assembled	global matrix are available on all process
matrix = distributedglobal	the global matrix is distributed among all the process
matrix = distributed	the input matrix is distributed (not yet implemented)

The option arguments of SuperLU_DIST are described in the section Users-callable routine of [?]. The parameter Fact and TRANS are specified in FreeFem++ interfaces to SuperLU_DIST during the different steps. For this reason, the value given by the user for this option is not considered. The factorization LU is calculated in SuperLU_DIST on the matrix A_p .

$$A_p = P_c P_r D_r A D_c P_c^t$$

where P_c and P_r is the row and column permutation matrix respectively, D_r and D_c are diagonal matrix for respectively row and column scaling. The option argument RowPerm (resp. ColPerm) control the row (resp. column) permutation matrix. D_r and D_c is controlled by the parameter DiagScale. The parameter permr, permc, scaler, and scalec in FreeFem++ is provided to give row permutation, column permutation, row scaling and column scaling of the user respectively. The other parameters for LU factorization are ParSymFact and ReplaceTinyPivot. The parallel symbolic factorization works only on a power of two processes and need the ParMetis ordering [?]. The default option argument of SuperLU_DIST are given in the file `ffsuperlu_dist_fileparam.txt`.

Calling SuperLU_DIST in FreeFem++ To call SuperLU_DIST in FreeFem++ , we need to load the library dynamic correspond to interface. This done by the following line **load "real_superlu_DIST_FreeFem"** (resp. **load "complex_superlu_DIST_FreeFem"**) for real (resp. complex) arithmetics in the file `.edp`.

Solver parameters is defined in .edp file: To call SuperLU_DIST with internal parameter, we used the parameters `sparams`. The value of parameters of SuperLU_DIST in `sparams` is defined by

```
sparams  ="nprow=1, npcol=1, matrix= distributedglobal, Fact= DOFACT, Equil=NO,
          ParSymbFact=NO, ColPerm= MMD_AT_PLUS_A, RowPerm= LargeDiag,
          DiagPivotThresh=1.0, IterRefine=DOUBLE, Trans=NOTRANS,
          ReplaceTinyPivot=NO, SolveInitialized=NO, PrintStat=NO, DiagScale=NOEQUIL "
```

This value correspond to the parameter in the file `ffsuperlu_dist_fileparam.txt`. If one parameter is not specify by the user, we take the default value of SuperLU_DIST.

Reading solver parameters on a file: The structure of data file for SuperLU_DIST in FreeFem++ is given in the file `ffsuperlu_dist_fileparam.txt` (default value of the FreeFem++ interface).

```
1                      /* nprow : integer value      */
1                      /* npcol : integer value      */
distributedglobal      /* matrix input : assembled, distributedglobal, distributed
*/
DOFACT                 /* Fact   : DOFACT, SamePattern, SamePattern_SameRowPerm,
FACTORED */
NO                     /* Equil  : NO, YES */
NO                     /* ParSymbFact : NO, YES */
MMD_AT_PLUS_A         /* ColPerm : NATURAL, MMD_AT_PLUS_A, MMD_ATA, METIS_AT_PLUS_A, PARMETIS,
MY_PERMC */
LargeDiag              /* RowPerm : NOROWPERM, LargeDiag, MY_PERMR */
1.0                    /* DiagPivotThresh : real value */
DOUBLE                 /* IterRefine : NOREFINE, SINGLE, DOUBLE, EXTRA */
NOTRANS                /* Trans    : NOTRANS, TRANS, CONJ*/
NO                     /* ReplaceTinyPivot : NO, YES*/
NO                     /* SolveInitialized : NO, YES*/
NO                     /* RefineInitialized : NO, YES*/
NO                     /* PrintStat : NO, YES*/
NOEQUIL                /* DiagScale : NOEQUIL, ROW, COL, BOTH*/
```

If no solver parameter is given, we used default option of SuperLU_DIST solver.

Example 11.2 *A simple example of calling SuperLU_DIST in FreeFem++ with this two methods is given in the file `testsolver-superLU_DIST.edp` in the directory `examples++-mpi`.*

11.2.3 Pastix solver

Pastix (Parallel Sparse matrix package) is a free scientific library under CECILL-C license. This package solves sparse linear system with a direct and block ILU(k) iterative methods. This solver can be applied to a real or complex matrix with a symmetric pattern [?].

Installation of Pastix: To used Pastix in FreeFem++ , you have to install pastix package in first. To compile this package, we need a fortran 90 compiler, scotch [?] or Metis [?] ordering library and MPI. An installation procedure to obtain this package is given in the file `.src/solver/README_COMPILE` in the section pastix of the FreeFem++ package.

Creating Library of pastix interface for FreeFem++ : The FreeFem++ interface to pastix is given in file `real_pastix_FreeFem.cpp` (resp. `complex_pastix_FreeFem.cpp`) for real (resp. complex) arithmetics. This interface is compatible with the release 2200 of pastix and is designed for a global matrix. We have also implemented interface for distributed matrices. To use pastix in FreeFem++ , we need the library corresponding to this interface. A description to obtain this library is given in the file `README_COMPILE` in the directory `src/solver` of FreeFem++ . We recall here the procedure. Go to the directory `src/solver` in FreeFem++ package. Edit the file `makefile-sparsesolver.inc` to yours system : comment Section 1, comment line corresponding to libraries BLAS, METIS and SCOTCH in Section 2 and comment in Section 3 the paragraph corresponding to pastix solver. And just type **make rpastix** (resp. **make cpastix**) in the terminal to obtain the dynamic library of interface for real (resp. complex) arithmetics. Now we give a short description of pastix parameters before describing the method to call pastix in FreeFem++ .

Pastix parameters: The input matrix parameter of FreeFem++ depend on pastix interface. `matrix=assembled` for non distributed matrix. It is the same parameter for SuperLU_DIST. There are four parameters in Pastix : `iparm`, `dparm`, `perm` and `invp`. These parameters are respectively the integer parameters (vector of size 64), real parameters (vector of size 64), permutation matrix and inverse permutation matrix respectively. `iparm` and `dparm` vectors are described in [?]. The parameters `permr` and `permc` in FreeFem++ are provided to give permutation matrix and inverse permutation matrix of the user respectively.

Solver parameters defined in .edp file: To call Pastix in FreeFem++ in this case, we need to specify the parameters **lparams** and **dparams**. These parameters are defined by

$$\forall i = 0, \dots, 63, \quad \text{lparams}[i] = \text{iparm}[i].$$

$$\forall i = 0, \dots, 63, \quad \text{dparams}[i] = \text{dparm}[i].$$

Reading solver parameters on a file: The structure of data file for pastix parameters in FreeFem++ is : first line structure parameters of the matrix and in the following line the value of vectors `iparm` and `dparm` in this order.

```
assembled /* matrix input :: assembled, distributed global and distributed */
iparm[0]
iparm[1]
...
...
iparm[63]
dparm[0]
dparm[1]
...
...
dparm[63]
```

An example of this file parameter is given in `ffpastix_iparm_dparm.txt` with a description of these parameters. This file is obtained with the example file `iparm.txt` and `dparm.txt` including in the

direct solver	square matrix			rectangular matrix			
	sym	sym pattern	unsym	sym	sym pattern	unsym	
SuperLU_DIST	yes	yes	yes	yes	yes	yes	
MUMPS	yes	yes	yes	no	no	no	
pastix	yes	yes	no	no	no	no	

Table 11.3: Type of matrix used by the different direct sparse solver

pastix package.

If no solver parameter is given, we use the default option of pastix solver.

Example: A simple example of calling pastix in FreeFem++ with this two methods is given in the file testsolver.pastix.edp in the directory examples++-mpi.

In Table 11.3, we recall the different matrix considering in the different direct solvers.

11.3 Parallel sparse iterative solver

Concerning **iterative solvers**, we have chosen *pARMS* [?] , *HIPS* [?] and *Hypre* [?]. Each software implements a different type of parallel preconditioner. So, *pARMS* implements algebraic domain decomposition preconditioner type such as additive Schwartz [?] and interface method [?]; while HIPS implement hierarchical incomplete factorization [?] and finally HYPRE implements multilevel preconditioner are AMG(Algebraic MultiGrid) [?] and parallel approximated inverse [?].

To use one of these programs in FreeFem++ , you have to install it independently of FreeFem++ . It is also necessary to install the MPI communication library which is essential for communication between the processors and, in some cases, software partitioning graphs like METIS [?] or Scotch [?].

All this preconditioners are used with Krylov subspace methods accelerators. Krylov subspace methods are iterative methods which consist in finding a solution x of linear system $Ax = b$ inside the affine space $x_0 + K_m$ by imposing that $b - Ax \perp \mathcal{L}_m$, where K_m is Krylov subspace of dimension m defined by $K_m = \{r_0, Ar_0, A^2r_0, \dots, A^{m-1}r_0\}$ and \mathcal{L}_m is another subspace of dimension m which depends on type of Krylov subspace. For example in GMRES, $\mathcal{L}_m = AK_m$.

We realized an interface which is easy to use, so that the call of these different softwares in FreeFem++ is done in the same way. You just have to load the solver and then specify the parameters to apply to the specific solvers. In the rest of this chapter, when we talk about Krylov subspace methods we mean one among GMRES, CG and BICGSTAB.

11.3.1 pARMS solver

pARMS (*parallel Algebraic Multilevel Solver*) is a software developed by Youssef Saad and al at University of Minnesota [?]. This software is specialized in the resolution of large sparse non symmetric linear systems of equation. Solvers developed in pARMS is the Krylov subspace type. It consists of variants of GMRES like FGMRES(Flexible GMRES) , DGMRES(Deflated GMRES) [?] and BICGSTAB. pARMS also implements parallel preconditioner like RAS (Restricted Additive Schwarz)[?] and Schur Complement type preconditioner [?].

All these parallel preconditioners are based on the principle of domain decomposition. Thus, the matrix A is partitioned into sub matrices $A_i (i = 1, \dots, p)$ where p represents the number of partitions one needs. The union of A_i forms the original matrix. The solution of the overall system is obtained by solving the local systems on A_i (see [?]). Therefore, a distinction is made between iterations on A and the local iterations on A_i . To solve the local problem on A_i there are several preconditioners as **ilut** (Incomplete LU with threshold), **iluk** (Incomplete LU with level of fill in) and **ARMS** (Algebraic Recursive Multilevel Solver). But to use pAMRS in FreeFem++ you have first to install pAMRS.

Installation of pARMS To install pARMS, you must first download the pARMS package at [?]. Once the download is complete, you must unpack package pARMS and follow the installation procedure described in file README to create the library **libparms.a**.

Using pARMS as interface to FreeFem++ Before calling pARMS solver inside FreeFem++ , you must compile file *parms_FreeFem.cpp* to create a dynamic library *parms_FreeFem.so*. To do this, move to the directory *src/solver* of FreeFem++ , edit the file *makefileparms.inc* to specify the following variables:

<i>PARMS_DIR</i> :	Directory of pARMS
<i>PARMS_INCLUDE</i> :	Directory for header of pARMS
<i>METIS</i> :	METIS directory
<i>METIS_LIB</i> :	METIS librairy
<i>MPI</i> :	MPI directory
<i>MPI_INCLUDE</i> :	MPI headers
<i>FREEFEM</i> :	FreeFem++ directory
<i>FREEFEM_INCLUDE</i> :	FreeFem++ header for sparse linear solver
<i>LIBBLAS</i> :	Blas library

After that, in the command line type **make parms** to create *parms_FreeFem.so*.

As usual in FreeFem++ , we will show by examples how to call pARMS in FreeFem++ . There are three ways of doing this:

Example 1: Default parameters This example comes from user guide of FreeFem++ [?] at page 12.

Example 11.3

```

1: load parms_freemem // Tell FreeFem that you will use pARMS
2: border C(t=0,2*pi){x=cos(t); y=sin(t);label=1;}
3: mesh Th = buildmesh (C(50));
4: fespace Vh(Th,P2);
5: Vh u,v;
6: func f= x*y;
7: problem Poisson(u,v,solver=sparsecv) = // bilinear part will use
8:   int2d(Th)(dx(u)*dx(v) + dy(u)*dy(v)) // a sparse solver, in this
case pARMS
9:   - int2d(Th)( f*v) // right hand side
10:   + on(1,u=0) ; // Dirichlet boundary condition
11:
12: real cpu=clock();
13: Poisson; // SOLVE THE PDE
14: plot(u);
15: cout << " CPU time = " << clock()-cpu << endl;
```


In line 1 of example 11.3 we load in memory the pARMS dynamic library with interface FreeFem++. After this, in line 7 we specify that the bilinear form will be solved by the last sparse linear solver load in memory which, in this case, is pARMS.

The parameter used in pARMS in this case is the default one since the user does not have to provide any parameter.

Here are some default parameters:

solver=FGMRES, Krylov dimension=30, Maximum of Krylov=1000, Tolerance for convergence=1e-08.(see book of Saad [?] to understand all this parameters.)

preconditionner=Restricted Additif Schwarz [?], Inner Krylov dimension=5, Maximum of inner Krylov dimension=5, Inner preconditionner=ILUK.

To specify the parameters to apply to the solver, the user can either give an integer vector for **integer parameters** and real vectors for **real parameters** or provide a **file** which contains those parameters.

Example 2: User specifies parameters inside two vectors Lets us consider Navier Stokes example 11.4 . In this example we solve linear systems coming from discretization of Navier Stokes equation with pARMS. Parameters of solver is specified by user.

```

Example 11.4 (Stokes.edp) include "manual.edp"
include "includes.edp";
include "mesh_with_cylinder.edp";
include "bc_poiseuille_in_square.edp";
include "fe_functions.edp";
0: load parms_FreeFem
1: int[int] iparm(16); real[int] dparm(6);
2: int ,ii;
3: for(ii=0;ii<16;ii++){iparm[ii]=-1;} for(ii=0;ii<6;ii++) dparm[ii]=-1.0;
4: fespace Vh(Th, [P2,P2,P1]);
5: iparm[0]=0;
6: varf Stokes ([u,v,p],[ush,vsh,psh],{solver=sparsesolver}) =
    int2d(Th) ( nu*( dx(u)*dx(ush) + dy(u)*dy(ush) + dx(v)*dx(vsh) + dy(v)*dy(vsh) )
              - p*psh*(1.e-6) // p epsilon
              - p*(dx(ush)+dy(vsh)) // + dx(p)*ush + dy(p)*vsh
              - (dx(u)+dy(v))*psh // psh div(u)
    )
    + on(cylinder,infwall,supwall,u=0.,v=0.)+on(inlet,u=uc,v=0); // Bdy
conditions
7: matrix AA=Stokes(VVh,VVh);
8: set (AA,solver=sparsesolver,lparams=iparm,dparams=dparm); // Set pARMS as
linear solver
9: real[int] bb= Stokes(0,VVh); real[int] sol(AA.n);
10: sol= AA^-1 * bb;

```

We need two vectors to specify the parameters of the linear solver. In line 1 of example 11.4 we have declared these vectors(**int**[**int**] **iparm**(16); **real**[**int**] **dparm**(6);) . In line 3 we have initialized these vectors by negative values. We do this because all parameters values in pARMS are positive and if you do not change the negative values of one entry of this vector, the default value will be set. In tables (table 11.4 and 11.5) , we have the meaning of differents entries of these vectors.

We run example 11.4 on cluster paradipt of Grid5000 and report results in table 11.8.

Entries of iparm	Significations of each entries
iparm[0]	Krylov subspace methods. Differents values for this parameters are specify on table 11.6
iparm[1]	Preconditionner. Differents preconditionners for this parameters are specify on table 11.7
iparm[2]	Krylov subspace dimension in outer iteration: default value 30
iparm[3]	Maximum of iterations in outer iteration: default value 1000
iparm[4]	Number of level in arms when used.
iparm[5]	Krylov subspace dimension in inner iteration: default value 3
iparm[6]	Maximum of iterations in inner iteration: default value 3
iparm[7]	Symmetric(=1 for symmetric) or unsymmetric matrix: default value 0(unsymmetric matrix)
iparm[8]	Overlap size between different subdomain: default value 0(no overlap)
iparm[9]	Scale the input matrix or not: Default value 1 (Matrix should be scale)
iparm[10]	Block size in arms when used: default value 20
iparm[11]	lfl0 (ilut, iluk, and arms) : default value 20
iparm[12]	lfl for Schur complement const : default value 20
iparm[13]	lfl for Schur complement const : default value 20
iparm[14]	Multicoloring or not in ILU when used : default value 1
iparm[15]	Inner iteration : default value 0
iparm[16]	Print message when solving:default 0(no message print). 0: no message is print, 1: Convergence informations like number of iteration and residual , 2: Timing for a different step like preconditioner 3 : Print all informations.

Table 11.4: Meaning of **lparams** corresponding variables for example 11.4

Entries of dparam	Significations of each entries
dparam[0]	precision for outer iteration : default value 1e-08
dparam[1]	precision for inner iteration: default value 1e-2
dparam[2]	tolerance used for diagonal domain: : default value 0.1
dparam[3]	drop tolerance droptol0 (ilut, iluk, and arms) : default value 1e-2
dparam[4]	droptol for Schur complement const: default value 1e-2
dparam[5]	droptol for Schur complement const: default value 1e-2

Table 11.5: Significations of **dparams** corresponding variables for example 11.4

Values of iparm[0]	Krylov subspace methods
0	FGMRES (Flexible GMRES)
1	DGMRES (Deflated GMRES)
2	BICGSTAB

Table 11.6: Krylov Solvers in pARMS

Values of iparm[1]	Preconditionners
0	Preconditioners type is <i>additive Schwartz preconditioner with ilu0 as local preconditioner,</i>
1	preconditioner type is <i>additive Schwartz preconditioner with iluk as local preconditioner,</i>
2	preconditioner type is <i>additive Schwartz preconditioner with ilut as local preconditioner,</i>
3	preconditioner type is <i>additive Schwartz preconditioner with arms as local preconditioner,</i>
4	preconditioner type is <i>Left Schur complement preconditioner with ilu0 as local preconditioner,</i>
5	preconditioner type is <i>Left Schur complement preconditioner with ilut as local preconditioner,</i>
6	preconditioner type is <i>Left Schur complement preconditioner with iluk as local preconditioner,</i>
7	preconditioner type is <i>Left Schur complement preconditioner with arms as local preconditioner,</i>
8	preconditioner type is <i>Right Schur complement preconditioner with ilu0 as local preconditioner,</i>
9	preconditioner type is <i>Right Schur complement preconditioner with ilut as local preconditioner,</i>
10	preconditioner type is <i>Right Schur complement preconditioner with iluk as local preconditioner,</i>
11	preconditioner type is <i>Right Schur complement preconditioner with arms as local preconditioner,</i>
12	preconditioner type is <i>sch_gilu0</i> , Schur complement preconditioner with global ilu0
13	preconditioner type is <i>SchurSymmetric GS preconditioner</i>

Table 11.7: Preconditionners in pARMS

n= 471281		nnz=13×10^6		Te=571,29
np	add(iluk)		schur(iluk)	
	nit	time	nit	time
4	230	637.57	21	557.8
8	240	364.12	22	302.25
16	247	212.07	24	167.5
32	261	111.16	25	81.5

Table 11.8: Convergence and time for solving linear system from example 11.4

n	matrix size
nnz	number of non null entries inside matrix
nit	number of iteration for convergence
time	Time for convergence
Te	Time for constructing finite element matrix
np	number of processor

Table 11.9: Legend of table 11.8

In this example, we fix the matrix size (in term of finite element, we fix the mesh) and increase the number of processors used to solve the linear system. We saw that, when the number of processors increases, the time for solving the linear equation decreases, even if the number of iteration increases. This proves that, using pARMS as solver of linear systems coming from discretization of partial differential equation in FreeFem++ can decrease drastically the total time of simulation.

11.3.2 Interfacing with HIPS

HIPS (*Hierarchical Iterative Parallel Solver*) is a scientific library that provides an efficient parallel iterative solver for very large sparse linear systems. HIPS is available as free software under the CeCILL-C licence. The interface that we realized is compatible with release **1.2 beta.rc4** of HIPS. HIPS implements two solver classes which are the iteratives class (GMRES, PCG) and the Direct class. Concerning preconditionners, HIPS implements a type of multilevel ILU. For further informations on those preconditionners see [?, ?].

Installation of HIPS To install HIPS, first download the HIPS package at [?], unpack it and go to the HIPS source directory. The installation of HIPS is machine dependence. For example, to install HIPS on a linux cluster copy the file *Makefile_Inc_Examples/makefile.inc.gnu* on the root directory of HIPS with the name **makefile.inc**. After this, edit **makefile.inc** to set values of different variables and type **make all**.

Using HIPS as the interface to FreeFem++ Before calling the HIPS solver inside FreeFem++, you must compile file *hips_FreeFem.cpp* to create dynamic library *hips_FreeFem.so*. To do this, move to the directory *src/solver* of FreeFem++ and edit the file *makefile.inc* to specify the following variables:

<i>HIPS_DIR</i> :	Directory of HIPS
<i>HIPS_INCLUDE</i> :	-I\$(<i>HIPS_DIR</i>)/SRC/INCLUDE : Directory for HIPS headers
<i>LIB_DIR</i> :	-L\$(<i>HIPS_DIR</i>)/LIB : Libraries directory
<i>LIBHIPSSEQUENTIAL</i> :	\$(<i>HIPS_DIR</i>)/LIB/libhipssequential.a: HIPS utilities library
<i>LIBHIPS</i> :	\$(<i>HIPS_DIR</i>)/LIB/libhips.a: HIPS library
<i>FREEFEM</i> :	FreeFem++ directory
<i>FREEFEM_INCLUDE</i> :	FreeFem headers for sparse linear solver
<i>METIS</i> :	METIS directory
<i>METIS_LIB</i> :	METIS library
<i>MPI</i> :	MPI directory
<i>MPI_INCLUDE</i> :	MPI headers

After specifies all the variables, in the command line in the directory *src/solver* type **make hips** to create *hips_FreeFem.so*.

Like with pARMS, the calling of HIPS in FreeFem++ can be done in three different manners. We will present only one example where the user specifies the parameters through keywords `lparams` and `dparams`.

Laplacian 3D solve with HIPS Let us consider the 3D Laplacian example inside FreeFem++ package where after discretization we want to solve the linear equation with Hips. Example 11.5 is Laplacian3D using Hips as linear solver. We first load Hips solver at line 2. From line 4 to 15 we specify the parameters for the Hips solver and in line 46 of example 11.5 we set these parameters in the linear solver.

In Table 11.10 results of running example 11.5 on Cluster Paradent of Grid5000 are reported. We can see in this running example the efficiency of parallelism.

Example 11.5 (Laplacian3D.edp) 1: `load "msh3"`

```

2: load "hips_FreeFem" // load library
3: int nn=10,iii;
4: int[int] iparm(14);
5: real[int] dparm(6);
6: for(iii=0;iii<14;iii++) iparm[iii]=-1;
7: for(iii=0;iii<6;iii++) dparm[iii]=-1;
8: iparm[0]=0; // use iterative solver
9: iparm[1]=1; // PCG as Krylov method
10: iparm[4]=0; // Matrix are symmetric
11: iparm[5]=1; // Pattern are also symmetric
12: iparm[9]=1; // Scale matrix
13: dparm[0]=1e-13; // Tolerance to convergence
14: dparm[1]=5e-4; // Threshold in ILUT
15: dparm[2]=5e-4; // Threshold for Schur preconditionner
16: mesh Th2=square(nn,nn);
17: fespace Vh2(Th2,P2);
18: Vh2 ux,uz,p2;
19: int[int] rup=[0,2], rdown=[0,1], rmid=[1,1,2,1,3,1,4,1];
20: real zmin=0,zmax=1;
21: mesh3 Th=buildlayers(Th2,nn,
    zbound=[zmin,zmax],
    reffacemid=rmid,
    reffaceup = rup,
    reffacelow = rdown);
22: savemesh(Th,"copie.mesh");
23: mesh3 Th3("copie.mesh");
24: fespace Vh(Th,P2);
25: func ue = 2*x*x + 3*y*y + 4*z*z + 5*x*y+6*x*z+1;
26: func uex= 4*x+ 5*y+6*z;
27: func uey= 6*y + 5*x;
28: func uez= 8*z +6*x;
29: func f= -18. ;
30: Vh uhe = ue; //
31: cout << " uhe min: " << uhe[].min << " max:" << uhe[].max << endl;
32: Vh u,v;
33: macro Grad3(u) [dx(u),dy(u),dz(u)] // EOM
34: varf va(u,v)= int3d(Th) (Grad3(v)' *Grad3(u)) // ') for emacs
    + int2d(Th,2) (u*v)
    - int3d(Th) (f*v)
    - int2d(Th,2) ( ue*v + (uex*N.x +uey*N.y +uez*N.z)*v )
    + on(1,u=ue);

```

$n = 4 \times 10^6$	$nnz = 118 \times 10^6$	Te=221.34
np	nit	time
8	190	120.34
16	189	61.08
32	186	31.70
64	183	23.44

Table 11.10: Iterations and Timing of solving linear system from example 11.5

```

35: real cpu=clock();
36: matrix Aa;
37: Aa=va(Vh,Vh);
38: varf l(unused,v)=int3d(Th)(f*v);
39: Vh F; F[]=va(0,Vh);
40: if(mpirank==0){
    cout << "Taille " << Aa.n << endl;
    cout << "Non zeros " << Aa.nbcoef << endl;
}
41: if(mpirank==0)
42:   cout << "CPU TIME FOR FORMING MATRIX = " << clock()-cpu << endl;
43:   set(Aa,solver=sparse solver,dparams=dparm,lparams=iparm); // Set hips
as linear solver
44:   u[]=Aa^-1*F[];

```

Legend of table 11.10 are give in table 11.9.

11.3.3 Interfacing with HYPRE

HYPRE (*High Level Preconditioner*) is a suite of parallel preconditioner developed at Lawrence Livermore National Lab [?].

There are two main classes of preconditioners developed in HYPRE: AMG (Algebraic MultiGrid) and Parasails (Parallel Sparse Approximate Inverse).

Now, suppose we want to solve $Ax = b$. At the heart of AMG there is a series of progressively coarser(smaller) representations of the matrix A . Given an approximation \hat{x} to the solution x , consider solving the residual equation $Ae = r$ to find the error e , where $r = b - A\hat{x}$. A fundamental principle of AMG is that it is an algebraically smooth error. To reduce the algebraically smooth errors further, they need to be represented by a smaller defect equation (coarse grid residual equation) $A_c e_c = r_c$, which is cheaper to solve. After solving this coarse equation, the solution is then interpolated in fine grid represented here by matrix A . The quality of AMG depends on the choice of coarsening and interpolating operators.

The *sparse approximate inverse* approximates the inverse of a matrix A by a sparse matrix M . A technical idea to construct matrix M is to minimize the Frobenius norm of the residual matrix $I - MA$. For more details on this preconditioner technics see [?].

HYPRE implement three Krylov subspace solvers: GMRES, PCG and BiCGStab.

Installation of HYPRE To install HYPRE, first download the HYPRE package at [?], unpack it and go to the HYPRE/src source directory and do **./configure** to configure Hypre. After this just type **make all** to create **libHYPRE.a**.

Entries of iparm	Significations of each entries
iparm [0]	Strategy use for solving (Iterative=0 or Hybrid=1 or Direct=2). Defaults values are : Iterative
iparm [1]	Krylov methods. If iparm [0]=0, give type of Krylov methods: 0 for GMRES, 1 for PCG
iparm [2]	Maximum of iterations in outer iteration: default value 1000
iparm [3]	Krylov subspace dimension in outer iteration: default value 40
iparm [4]	Symmetric(=0 for symmetric) and 1 for unsymmetric matrix: default value 1(unsymmetric matrix)
iparm [5]	Pattern of matrix are symmetric or not: default value 0
iparm [6]	Partition type of input matrix: default value 0
iparm [7]	Number of level that use the HIPS locally consistent fill-in: Default value 2
iparm [8]	Numbering in indices array will start at 0 or 1: Default value 0
iparm [9]	Scale matrix. Default value 1
iparm [10]	Reordering use inside subdomains for reducing fill-in: Only use for iterative. Default value 1
iparm [11]	Number of unknowns per node in the matrix non-zero pattern graph: Default value 1
iparm [12]	This value is used to set the number of time the normalization is applied to the matrix: Default 2.
iparm [13]	Level of informations printed during solving: Default 5.
iparm [14]	HIPS_DOMSIZE Subdomain size

Table 11.11: Significations of **lparams** corresponding to HIPS interface

dparm [0]	<i>HIPS_PREC</i> : Relative residual norm: Default=1e-9
dparm [1]	<i>HIPS_DROPTOL0</i> : Numerical threshold in ILUT for interior domain (important : set 0.0 in HYBRID: Default=0.005)
dparm [2]	<i>HIPS_DROPTOL1</i> : Numerical threshold in ILUT for Schur preconditioner: Default=0.005
dparm [3]	<i>HIPS_DROPTOLE</i> : Numerical threshold for coupling between the interior level and Schur: Default 0.005
dparm [4]	<i>HIPS_AMALG</i> : Numerical threshold for coupling between the interior level and Schur: Default=0.005
dparm [5]	<i>HIPS_DROPSCHUR</i> : Numerical threshold for coupling between the interior level and Schur: Default=0.005

Table 11.12: Significations of **dparams** corresponding to HIPS interface


```
43:      set (Aa, solver=sparse solver, dparams=dparm, lparams=iparm);
```

iparms[0]	Solver identification: 0: BiCGStab, 1: GMRES, 2: PCG. By default=1
iparms[1]	Preconditioner identification: 0: BOOMER AMG, 1: PILUT, 2: Parasails, 3: Schwartz Default=0
iparms[2]	Maximum of iteration: Default=1000
iparms[3]	Krylov subspace dim: Default= 40
iparms[4]	Solver print info level: Default=2
iparms[5]	Solver log : Default=1
iparms[6]	Solver stopping criteria only for BiCGStab : Default=1
dparams[0]	Tolerance for convergence : <i>Default</i> = $1.0e - 11$

Table 11.13: Definitions of common entries of **iparms** and **dparams** vectors for every preconditioner in HYPRE

11.3.4 Conclusion

With the different runs presented here, we wanted to illustrate the gain in time when we increase the number of processors used for the simulations. We saw that in every case the time for the construction of the finite element matrix is constant. This is normal because until now this phase is sequential in FreeFem++ . In contrast, phases for solving the linear system are parallel. We saw on several examples presented here that when we increase the number of processors, in general we decrease the time used for solving the linear systems. But this not true in every case. In several case, when we increase the number of processors the time to convergence also increases. There are two main reasons for this. First, the increase of processors can lead to the increase of volume of exchanged data across processors consequently increasing the time for solving the linear systems. Furthermore, in decomposition domain type preconditioners, the number of processors generally corresponds to the number of sub domains. In subdomain methods, generally when we increase the number of subdomains we decrease convergence quality of the preconditioner. This can increase the time used for solving linear equations.

To end this, we should note that good use of the preconditioners interfaced in FreeFem++ is empiric, because it is difficult to know what is a good preconditioner for some type of problems. Although, the efficiency of preconditioners sometimes depends on how its parameters are set. For this reason we advise the user to pay attention to the meaning of the parameters in the user guide of the iterative solvers interfaced in FreeFem++ .

11.4 Domain decomposition

In the previous section, we saw that the phases to construct a matrix are sequential. One strategy to construct the matrix in parallel is to divide geometrically the domain into subdomains. In every subdomain we construct a local submatrix and after that we assemble every submatrix to form the global matrix.

We can use this technique to solve pde directly in domain Ω . In this case, in every subdomains you have to define artificial boundary conditions to form consistent equations in every subdomains. After this, you solve equation in every subdomains and define a strategy to obtain the global solution.

iparms[7]	AMG interpolation type: Default=6
iparms[8]	Specifies the use of GSMG - geometrically smooth coarsening and interpolation: Default=1
iparms[9]	AMG coarsen type: Default=6
iparms[10]	Defines whether local or global measures are used: Default=1
iparms[11]	AMG cycle type: Default=1
iparms[12]	AMG Smoother type: Default=1
iparms[13]	AMG number of levels for smoothers: Default=3
iparms[14]	AMG number of sweeps for smoothers: Default=2
iparms[15]	Maximum number of multigrid levels: Default=25
iparms[16]	Defines which variant of the Schwartz method is used: 0: hybrid multiplicative Schwartz method (no overlap across processor boundaries) 1: hybrid additive Schwartz method (no overlap across processor boundaries) 2: additive Schwartz method 3: hybrid multiplicative Schwartz method (with overlap across processor boundaries) Default=1
iparms[17]	Size of the system of PDEs: Default=1
iparms[18]	Overlap for the Schwarz method: Default=1
iparms[19]	Type of domain used for the Schwarz method 0: each point is a domain 1: each node is a domain (only of interest in “systems” AMG) 2: each domain is generated by agglomeration (default)
dparms[1]	AMG strength threshold: Default=0.25
dparms[2]	Truncation factor for the interpolation: Default=1e-2
dparms[3]	Sets a parameter to modify the definition of strength for diagonal dominant portions of the matrix: Default=0.9
dparms[3]	Defines a smoothing parameter for the additive Schwartz method Default=1.

Table 11.14: Definitions of other entries of **iparms** and **dparms** if preconditioner is **BOOMER AMG**

iparms[7]	Row size in Parallel ILUT: Default=1000
iparms[8]	Set maximum number of iterations: Default=30
dparms[1]	Drop tolerance in Parallel ILUT: Default=1e-5

Table 11.15: Definitions of other entries of **iparms** and **dparms** if preconditioner is **PILUT**

iparms[7]	Number of levels in Parallel Sparse Approximate inverse: Default=1
iparms[8]	Symmetric parameter for the ParaSails preconditioner: 0: nonsymmetric and/or indefinite problem, and nonsymmetric preconditioner 1: SPD problem, and SPD (factored) preconditioner 2: nonsymmetric, definite problem, and SPD (factored) preconditioner Default=0
dparms[1]	Filters parameters: The filter parameter is used to drop small nonzeros in the preconditioner, to reduce the cost of applying the preconditioner: Default=0.1
dparms[2]	Threshold parameter: Default=0.1

Table 11.16: Definitions of other entries of **iparms** and **dparms** if preconditioner is **ParaSails**

iparms[7]	Defines which variant of the Schwartz method is used: 0: hybrid multiplicative Schwartz method (no overlap across processor boundaries) 1: hybrid additive Schwartz method (no overlap across processor boundaries) 2: additive Schwartz method 3: hybrid multiplicative Schwartz method (with overlap across processor boundaries) Default=1
iparms[8]	Overlap for the Schwartz method: Default=1
iparms[9]	Type of domain used for the Schwartz method 0: each point is a domain 1: each node is a domain (only of interest in “systems” AMG) 2: each domain is generated by agglomeration (default)

Table 11.17: Definitions of other entries of **iparms** and **dparms** if preconditioner is **Schwartz**

n=4 × 10⁶	nnz=13 × 10⁶	Te=571,29
np	AMG	
	nit	time
8	6	1491.83
16	5	708.49
32	4	296.22
64	4	145.64

Table 11.18: Convergence and time for solving linear system from example 11.4

In terms of parallel programming for FreeFem++ , with MPI, this means that the user must be able to divide processors available for computation into subgroups of processors and also must be able to realize different type of communications in FreeFem++ script. Here is a wrapper of some MPI functions.

11.4.1 Communicators and groups

Groups

mpiGroup grpe(mpiGroup gp,KN_ < long >): Create *MPI_Group* from existing group **gp** by given vector

Communicators

Communicators is an abstract MPI object which allows MPI user to communicate across group of processors. Communicators can be Intracommunicators(involves a single group) or Intercommunicators (involves two groups). When we not specify type of communicator it will be Intracommunicators

mpiComm cc(mpiComm comm, mpiGroup gp): Creates a new communicator. *comm* communicator(handle), *gp* group which is a subset of the group of *comm* (handle). Return new communicator

mpiComm cc(mpiGroup gp): Same as previous constructor but default *comm* here is MPI_COMM_WORLD.

mpiComm cc(mpiComm comm, int color, int key): Creates new communicators based on *colors* and *key*. This constructor is based on MPI_Comm_split routine of MPI.

mpiComm cc(MPIrank p,int key): Same constructor than the last one. Here *colors* and *comm* is defined in *MPIrank*. This constructor is based on MPI_Comm_split routine of MPI.

Example 11.7 (commsplit.edp)

```
1: int color=mpiRank(comm)%2;
2: mpiComm ccc(processor(color,comm),0);
3: mpiComm qpp(comm,);
4: mpiComm cp(cc,color,0);
```

mpiComm cc(mpiComm comm, int high): Creates an intracommunicator from an intercommunicator. *comm* intercommunicator, *high* Used to order the groups within *comm* (logical) when creating the new communicator. This constructor is based on MPI_Intercomm_merge routine of MPI.

mpiComm cc(MPIrank p1, MPIrank p2, int tag): This constructor creates an intercommunicator from two intracommunicators. *p1* defined local (intra)communicator and rank in local_comm of leader (often 0) while *p2* defined remote communicator and rank in peer_comm of remote leader (often 0). *tag* Message tag to use in constructing intercommunicator. This constructor is based on MPI_Intercomm_create.

Example 11.8 (merge.edp)

```
1: mpiComm comm,cc;
2: int color=mpiRank(comm)%2;
3: int rk=mpiRank(comm);
4: int size=mpiSize(comm);
4: cout << "Color values " << color << endl;
5: mpiComm ccc(processor((rk<size/2),comm),rk);
6: mpiComm cp(cc,color,0);
7: int rleader;
8: if (rk == 0) { rleader = size/2; }
9: else if (rk == size/2) { rleader = 0; }
```

```

10: else { rleader = 3; }
11: mpiComm qqp (processor (0,ccc),processor (rleader,comm),12345);
12:int aaa=mpiSize(ccc);
13:cout << "number of processor" << aaa << endl;

```

11.4.2 Process

In FreeFem++ we wrap MPI process by function call **processor** which create internal FreeFem++ object call **MPIrank**. This mean that do not use **MPIrank** in FreeFem++ script.

processor(int rk): Keep process rank inside object **MPIrank**. Rank is inside MPI.COMM_WORLD.
processor(int rk, mpiComm cc) and **processor(mpiComm cc,int rk)** process rank inside communicator cc.

processor(int rk, mpiComm cc) and **processor(mpiComm cc,int rk)** process rank inside communicator cc.

processorblock(int rk) : This function is exactly the same than **processor(int rk)** but is use in case of blocking communication.

processorblock(int rk, mpiComm cc) : This function is exactly the same than **processor(int rk,mpiComm cc)** but use a synchronization point.

11.4.3 Points to Points communicators

In FreeFem++ you can call MPI points to points communications functions.

Send(processor(int rk,mpiComm cc),Data D) : Blocking send of *Data D* to processor of rank *rk* inside communicator *cc*. Note that *Data D* can be: *int, real,complex, int[int], real[int],complex[int], Mesh, Mesh3, Matrix*.

Recv(processor(int rk,mpiComm cc),Data D): Receive *Data D* from process of rank *rk* in communicator *cc*. Note that *Data D* can be: *int, real,complex, int[int], real[int],complex[int], Mesh, Mesh3, Matrix* and should be the same type than corresponding send.

Isend(processor(int rk,mpiComm cc),Data D) : Non blocking send of *Data D* to processor of rank *rk* inside communicator *cc*. Note that *Data D* can be: *int, real,complex, int[int], real[int],complex[int], Mesh, Mesh3, Matrix*.

Recv(processor(int rk,mpiComm cc),Data D): Receive corresponding to send.

11.4.4 Global operations

In FreeFem++ you can call MPI global communication functions.

broadcast(processor(int rk,mpiComm cc),Data D): Process *rk* Broadcast *Data D* to all process inside communicator *cc*. Note that *Data D* can be: *int, real,complex, int[int], real[int],complex[int], Mesh, Mesh3, Matrix*.

broadcast(processor(int rk),Data D): Process *rk* Broadcast *Data D* to all process inside MPI.COMM_WORLD. Note that *Data D* can be: *int, real,complex, int[int], real[int],complex[int], Mesh, Mesh3, Matrix*.

mpiAlltoall(Data a,Data b): Sends *data a* from all to all processes. Receive buffer is *Data b*. This is done inside communicator MPI.COMM_WORLD.

mpiAlltoall(Data a, Data b, mpiComm cc): Sends *data a* from all to all processes. Receive buffer is *Data b*. This is done inside communicator *cc*.

mpiGather(Data a, Data b, processor(mpiComm, int rk) : Gathers together values *Data a* from a group of processes. Process of rank *rk* get data on communicator *rk*. This function is like MPI.Gather

mpiAllgather(Data a, Data b) : Gathers *Data a* from all processes and distribute it to all in *Data b*. This is done inside communicator MPI_COMM_WORLD. This function is like MPI.Allgather

mpiAllgather(Data a, Data b, mpiComm cc) : Gathers *Data a* from all processes and distribute it to all in *Data b*. This is done inside **communicator cc**. This function is like MPI.Allgather

mpiScatter(Data a, Data b, processor(int rk, mpiComm cc)) : Sends **Data a** from one process with rank **rk** to all other processes in group represented by communicator *mpiComm cc*.

mpiReduce(Data a, Data b, processor(int rk, mpiComm cc), MPI.Op op), Reduces values *Data a* on all processes to a single value *Data b* on process of rank *rk* and communicator *cc*. Operation use in reduce is: *MPI.Op op* which can be: *mpiMAX*, *mpiMIN*, *mpiSUM*, *mpiPROD*, *mpiLAND*, *mpiLOR*, *mpiLXOR*, *mpiBAND*, *mpiBXOR*, *mpiMAXLOC*, *mpiMINLOC*.

Note that, for all global operations, only int[int] and real[int] are data type take in account in FreeFem++ .

The following example present in details of Schwartz domain decomposition algorithm for solving Laplacian2d problem. In this example we use two level of parallelism to solve simple Laplacian2d in square domain. We have few number of subdomain and in every subdomain we use parallel sparse solver to solve local problem.

Example 11.9 (schwarz.edp)

```

1:load "hypr_FreeFem";           // Load Hypr solver
2:func bool AddLayers(mesh & Th,real[int] &ssd,int n,real[int] &unssd)
{
    // build a continuous function uunssd (P1) :
    // ssd in the characteristics function on the input sub domain.
    // such that :
    // unssd = 1 when ssd =1;
    // add n layer of element (size of the overlap)
    // and unssd = 0 outside of this layer ...
    // -----

    fespace Vh(Th,P1);
    fespace Ph(Th,P0);
    Ph s;
    assert(ssd.n==Ph.ndof);
    assert(unssd.n==Vh.ndof);
    unssd=0;
    s[] = ssd;

    // plot(s,wait=1,fill=1);

    Vh u;
    varf vM(u,v)=int2d(Th,qforder=1)(u*v/area);
    matrix M=vM(Ph,Vh);

    for(int i=0;i<n;++i)
    {
        u[] = M*s[];
    }
}

```

```

//      plot(u,wait=1);

u = u>.1;

//      plot(u,wait=1);

unssd+= u[];
s[] = M'*u[];
s = s >0.1;
}
unssd /= (n);
u[]=unssd;
ssd=s[];
return true;
}
3: mpiComm    myComm;           //      Create communicator with value MPI_COMM_WORLD

4: int membershipKey,rank,size;           //      Variables for manage communicators
5: rank=mpiRank(myComm); size=mpiSize(myComm); //      Rank of process and size
of communicator
6: bool withmetis=1, RAS=0;           //      Use or not metis for partitioning Mesh
7: int sizeoverlaps=5;           //      size off overlap
8: int withplot=1;
9: mesh Th=square(100,100);
10: int[int] chlab=[1,1 ,2,1 ,3,1 ,4,1 ];
11: Th=change(Th,refe=chlab);
12: int nn=2,mm=2, npart= nn*mm;
13: membershipKey = mpiRank(myComm)%npart; //      Coloring for partitioning
process group
14: mpiComm cc(processor(membershipKey,myComm),rank); //      Create MPI
communicator according previous coloring
15: fespace Ph(Th,P0), fespace Vh(Th,P1);
16: Ph part;
17: Vh sun=0,unssd=0;
18: real[int] vsum=sun[], reducesum=sun[]; //      Data use for control
partitioning.
19: Ph xx=x,yy=y,nupp;
20: part = int(xx*nn)*mm + int(yy*mm);
21: if(withmetis)
{
load "metis";
int[int] nupart(Th.nt);
metisdual(nupart,Th,npart);
for(int i=0;i<nupart.n;++i)
part[][i]=nupart[i];
}
22: if(withplot>1)
21: plot(part,fill=1,cmm="dual",wait=1);
22: mesh[int] aTh(npart);
23: mesh Thi=Th;
24: fespace Vhi(Thi,P1);
25: Vhi[int] au(npart),pun(npart);
26: matrix[int] Rih(npart), Dih(npart), aA(npart);
27: Vhi[int] auntgv(npart), rhsi(npart);
28: i=membershipKey;
Ph suppi= abs(part-i)<0.1;
AddLayers(Th,suppi[],sizeoverlaps,unssd[]);
Thi=aTh[i]=trunc(Th,suppi>0,label=10,split=1);
Rih[i]=interpolate(Vhi,Vh,inside=1); //      Vh -> Vhi

```

```

if(RAS)
{
    suppi= abs(part-i)<0.1;
    varf vSuppi(u,v)=int2d(Th,qforder=1)(suppi*v/area);
    unssd[]= vSuppi(0,Vh);
    unssd = unssd>0.;
    if(withplot>19)
        plot(unssd,wait=1);
}
pun[i][]=Rih[i]*unssd[];
sun[] += Rih[i]'*pun[i][]; // this is global operation
vsum=sun[]; // also global operation like broadcast';
if(withplot>9)
    plot(part,aTh[i],fill=1,wait=1);
    // Add mpireduce for sum all sun and pun local contribution.
29: mpiReduce(vsum, reducesum,processor(0,myComm),mpiSUM); // MPI global
operation MPI-Reduce on global communicator
30: broadcast(processor(0,myComm),reducesum); // Broadcast sum on process 0
to all process
31: sun[]=reducesum;
32: plot(sun,wait=1);
33: i=membershipKey
34: Thi=aTh[i];
35: pun[i]= pun[i]/sun;
36: if(withplot>8) plot(pun[i],wait=1);
37: macro Grad(u) [dx(u),dy(u)] // EOM
38: sun=0;
39: i=membershipKey
    Thi=aTh[i];
    varf va(u,v) =
        int2d(Thi)(Grad(u)'*Grad(v)) // ')
        +on(1,u=1) + int2d(Th)(v)
        +on(10,u=0) ;
40: aA[i]=va(Vhi,Vhi);
41: set(aA[i],solver=sparsesolver,mpicomm=cc); // Set parameters for Solver
Hypr. mpicomm=cc means you not solve on global process but in group on of process
define by cc
42: rhsi[i][]= va(0,Vhi);
43: Dih[i]=pun[i][];
44: real[int] un(Vhi.ndof);
45: un=1.;
46: real[int] ui=Dih[i]*un;
47: sun[] += Rih[i]'*ui; // ');
48: varf vaun(u,v) = on(10,u=1);
49: auntgv[i][]=vaun(0,Vhi); // store array of tgv on Gamma intern.
56: reducesum=0; vsum=sun;
57: mpiReduce(vsum, reducesum,processor(0,myComm),mpiSUM); // MPI global
operation MPI-Reduce on global communicator
58: broadcast(processor(0,myComm),reducesum); // Broadcast sum on process 0
to all other process
59: sun[]=reducesum;
60: if(withplot>5)
61: plot(sun,fill=1,wait=1);
62: cout << sun[].max << " " << sun[].min<< endl;
63: assert( 1.-1e-9 <= sun[].min && 1.+1e-9 >= sun[].max);
64: int nitermax=1000;

```



```

{
  Vh un=0;
  for(int iter=0;iter<nitermax;++iter)
  {
    real err=0,rerr=0;
    Vh un1=0;
    i=membershipKey;
    Thi=aTh[i];
    real[int] ui=Rih[i]*un[]; //  '
    real[int] bi = ui .* auntgv[i][];
    bi = auntgv[i][] ? bi : rhsi[i][];
    ui=au[i][];
    ui= aA[i] ^-1 * bi; // Solve local linear system on group of
    process represented by color membershipKey
    bi = ui-au[i][];
    err += bi'*bi; //  '
    au[i][]= ui;
    bi = Dih[i]*ui; // Prolongation of current solution to obtain
    right hand
    un1[] += Rih[i]*bi; //  '
  }
  65: reducesum=0; vsum=un1[];
  66: mpiReduce(vsum, reducesum,processor(0,myComm),mpiSUM); // MPI global
  operation MPi_Reduce on global communicator
  67: broadcast (processor(0,myComm),reducesum); // Broadcast sum on process 0
  to all other process
  68: un1[]=reducesum;
  69: real residrela=0;
  70: mpiReduce(err,residrela ,processor(0,myComm),mpiSUM);
  71: broadcast (processor(0,myComm),residrela);
  72: err=residrela; err= sqrt(err);
  73: if(rank==0) cout << iter << " Err = " << err << endl;
  74:   if(err<1e-5) break;
  75:   un[]=un1[];
  76:   if(withplot>2)
  77:     plot (au,dim=3,wait=0,cmm=" iter "+iter,fill=1 );
  78:   }
  79: plot (un,wait=1,dim=3);
  80: }

```


Chapter 12

Mesh Files

12.1 File mesh data structure

The mesh data structure, output of a mesh generation algorithm, refers to the geometric data structure and in some case to another mesh data structure.

In this case, the fields are

- MeshVersionFormatted 0
- Dimension (I) dim
- Vertices (I) NbOfVertices
 $\left(\left((R) x_i^j, j=1, \text{dim} \right), (I) Ref\phi_i^v, i=1, \text{NbOfVertices} \right)$
- Edges (I) NbOfEdges
 $\left(@@Vertex_i^1, @@Vertex_i^2, (I) Ref\phi_i^e, i=1, \text{NbOfEdges} \right)$
- Triangles (I) NbOfTriangles
 $\left(\left(@@Vertex_i^j, j=1, 3 \right), (I) Ref\phi_i^t, i=1, \text{NbOfTriangles} \right)$
- Quadrilaterals (I) NbOfQuadrilaterals
 $\left(\left(@@Vertex_i^j, j=1, 4 \right), (I) Ref\phi_i^t, i=1, \text{NbOfQuadrilaterals} \right)$
- Geometry
(C*) FileNameOfGeometricSupport
 - VertexOnGeometricVertex
(I) NbOfVertexOnGeometricVertex
 $\left(@@Vertex_i, @@Vertex_i^{geo}, i=1, \text{NbOfVertexOnGeometricVertex} \right)$
 - EdgeOnGeometricEdge
(I) NbOfEdgeOnGeometricEdge
 $\left(@@Edge_i, @@Edge_i^{geo}, i=1, \text{NbOfEdgeOnGeometricEdge} \right)$
- CrackedEdges (I) NbOfCrackedEdges
 $\left(@@Edge_i^1, @@Edge_i^2, i=1, \text{NbOfCrackedEdges} \right)$

When the current mesh refers to a previous mesh, we have in addition

- MeshSupportOfVertices
(C*) FileNameOfMeshSupport
- VertexOnSupportVertex
(I) NbOfVertexOnSupportVertex
(@@Vertex_i, @@Vertex_i^{supp}, i=1, NbOfVertexOnSupportVertex)
- VertexOnSupportEdge
(I) NbOfVertexOnSupportEdge
(@@Vertex_i, @@Edge_i^{supp}, (R) u_i^{supp}, i=1, NbOfVertexOnSupportEdge)
- VertexOnSupportTriangle
(I) NbOfVertexOnSupportTriangle
(@@Vertex_i, @@Tria_i^{supp}, (R) u_i^{supp}, (R) v_i^{supp},
i=1, NbOfVertexOnSupportTriangle)
- VertexOnSupportQuadrilaterals
(I) NbOfVertexOnSupportQuadrilaterals
(@@Vertex_i, @@Quad_i^{supp}, (R) u_i^{supp}, (R) v_i^{supp},
i=1, NbOfVertexOnSupportQuadrilaterals)

12.2 bb File type for Store Solutions

The file is formatted such that:

2 nbsol nbv 2

((U_{ij}, ∀i ∈ {1, ..., nbsol}), ∀j ∈ {1, ..., nbv})

where

- nbsol is a integer equal to the number of solutions.
- nbv is a integer equal to the number of vertex .
- U_{ij} is a real equal the value of the *i* solution at vertex *j* on the associated mesh background if read file, generated if write file.

12.3 BB File Type for Store Solutions

The file is formatted such that:

2 n typesol¹ ... typesolⁿ nbv 2
 (((U_{ij}^k, ∀i ∈ {1, ..., typesol^k}), ∀k ∈ {1, ..., n}) ∀j ∈ {1, ..., nbv})
 where

- n is a integer equal to the number of solutions
- typesol^k, type of the solution number *k*, is

- `typesolk = 1` the solution k is scalar (1 value per vertex)
- `typesolk = 2` the solution k is vectorial (2 values per unknown)
- `typesolk = 3` the solution k is a 2×2 symmetric matrix (3 values per vertex)
- `typesolk = 4` the solution k is a 2×2 matrix (4 values per vertex)
- `nbv` is a integer equal to the number of vertices
- U_{ij}^k is a real equal to the value of the component i of the solution k at vertex j on the associated mesh background if read file, generated if write file.

12.4 Metric File

A metric file can be of two types, isotropic or anisotropic.
the isotropic file is such that

`nbv 1`
`hi $\forall i \in \{1, \dots, \text{nbv}\}$`

where

- `nbv` is a integer equal to the number of vertices.
- h_i is the wanted mesh size near the vertex i on background mesh, the metric is $\mathcal{M}_i = h_i^{-2} Id$, where Id is the identity matrix.

The metric anisotrope

`nbv 3`
`a11i, a21i, a22i $\forall i \in \{1, \dots, \text{nbv}\}$`

where

- `nbv` is a integer equal to the number of vertices,
- $a11_i, a21_i, a22_i$ is metric $\mathcal{M}_i = \begin{pmatrix} a11_i & a21_i \\ a21_i & a22_i \end{pmatrix}$ which define the wanted mesh size in a vicinity of the vertex i such that h in direction $u \in \mathbb{R}^2$ is equal to $|u|/\sqrt{u \cdot \mathcal{M}_i u}$, where \cdot is the dot product in \mathbb{R}^2 , and $|\cdot|$ is the classical norm.

12.5 List of AM_FMT, AMDBA Meshes

The mesh is only composed of triangles and can be defined with the help of the following two integers and four arrays:

`nbv` is the number of vertices.

`nbv` is the number of vertices.

`nu(1:3, 1:nbv)` is an integer array giving the three vertex numbers
counterclockwise for each triangle.

`c(1:2,nbv)` is a real array giving the two coordinates of each vertex.
`refs(nbv)` is an integer array giving the reference numbers of the vertices.
`reft(nbv)` is an integer array giving the reference numbers of the triangles.

AM_FMT Files In fortran the `am_fmt` files are read as follows:

```
open(1, file='xxx.am_fmt', form='formatted', status='old')
  read (1, *) nbv, nbt
  read (1, *) ((nu(i, j), i=1, 3), j=1, nbt)
  read (1, *) ((c(i, j), i=1, 2), j=1, nbv)
  read (1, *) ( reft(i), i=1, nbt)
  read (1, *) ( refs(i), i=1, nbv)
close(1)
```

AM Files In fortran the `am` files are read as follows:

```
open(1, file='xxx.am', form='unformatted', status='old')
  read (1, *) nbv, nbt
  read (1) ((nu(i, j), i=1, 3), j=1, nbt),
& ((c(i, j), i=1, 2), j=1, nbv),
& ( reft(i), i=1, nbt),
& ( refs(i), i=1, nbv)
close(1)
```

AMDBA Files In fortran the `amdba` files are read as follows:

```
open(1, file='xxx.amdba', form='formatted', status='old')
  read (1, *) nbv, nbt
  read (1, *) (k, (c(i, k), i=1, 2), refs(k), j=1, nbv)
  read (1, *) (k, (nu(i, k), i=1, 3), reft(k), j=1, nbt)
close(1)
```

msh Files First, we add the notions of boundary edges

`nbbe` is the number of boundary edge.

`nube(1:2, 1:nbbe)` is an integer array giving the two vertex numbers

`refbe(1:nbbe)` is an integer array giving the two vertex numbers

In fortran the `msh` files are read as follows:

```
open(1, file='xxx.msh', form='formatted', status='old')
  read (1, *) nbv, nbt, nbbe
  read (1, *) ((c(i, k), i=1, 2), refs(k), j=1, nbv)
  read (1, *) ((nu(i, k), i=1, 3), reft(k), j=1, nbt)
  read (1, *) ((ne(i, k), i=1, 2), refbe(k), j=1, nbbe)
close(1)
```

ftq Files In fortran the ftq files are read as follows:

```
open(1,file='xxx.ftq',form='formatted',status='old')
read (1,*) nbv,nbe,nbt,nbq
read (1,*) (k(j),(nu(i,j),i=1,k(j)),reft(j),j=1,nbe)
read (1,*) ((c(i,k),i=1,2),refs(k),j=1,nbv)
close(1)
```

where if $k(j) = 3$ then the element j is a triangle and if $k = 4$ the the element j is a quadrilateral.

Chapter 13

Addition of a new finite element

13.1 Some notations

For a function \mathbf{f} taking value in \mathbb{R}^N , $N = 1, 2, \dots$, we define the finite element approximation $\Pi_h \mathbf{f}$ of \mathbf{f} . Let us denote the number of the degrees of freedom of the finite element by $NbDoF$. Then the i -th base ω_i^K ($i = 0, \dots, NbDoF - 1$) of the finite element space has the j -th component ω_{ij}^K for $j = 0, \dots, N - 1$.

The operator Π_h is called the interpolator of the finite element. We have the identity $\omega_i^K = \Pi_h \omega_i^K$. Formally, the interpolator Π_h is constructed by the following formula:

$$\Pi_h \mathbf{f} = \sum_{k=0}^{kPi-1} \alpha_k \mathbf{f}_{j_k}(P_{p_k}) \omega_{i_k}^K \quad (13.1)$$

where P_p is a set of $npPi$ points,

In the formula (13.1), the list p_k, j_k, i_k depend just on the type of finite element (not on the element), but the coefficient α_k can be depending on the element.

Example 1: with the classical scalar Lagrange finite element, we have $kPi = npPi = NbOfNode$ and

- P_p is the point of the nodal points
- the $\alpha_k = 1$, because we take the value of the function at the point P_k
- $p_k = k$, $j_k = k$ because we have one node per function.
- $j_k = 0$ because $N = 1$

Example 2: The Raviart-Thomas finite element:

$$RT0_h = \{\mathbf{v} \in H(div)/\forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y) = \left| \frac{\alpha_K}{\beta_K} + \gamma_K \begin{vmatrix} x \\ y \end{vmatrix} \right\} \quad (13.2)$$

The degrees of freedom are the flux through an edge e of the mesh, where the flux of the function $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is $\int_e \mathbf{f} \cdot \mathbf{n}_e$, \mathbf{n}_e is the unit normal of edge e (this implies a orientation of all the edges of the mesh, for example we can use the global numbering of the edge vertices and we just go to small to large number).

To compute this flux, we use a quadrature formula with one point, the middle point of the edge. Consider a triangle T with three vertices $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Let denote the vertices numbers by i_a, i_b, i_c , and define the three edge vectors $\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2$ by $\text{sgn}(i_b - i_c)(\mathbf{b} - \mathbf{c})$, $\text{sgn}(i_c - i_a)(\mathbf{c} - \mathbf{a})$, $\text{sgn}(i_a - i_b)(\mathbf{a} - \mathbf{b})$, The three basis functions are:

$$\omega_0^K = \frac{\text{sgn}(i_b - i_c)}{2|T|}(x - a), \quad \omega_1^K = \frac{\text{sgn}(i_c - i_a)}{2|T|}(x - b), \quad \omega_2^K = \frac{\text{sgn}(i_a - i_b)}{2|T|}(x - c), \quad (13.3)$$

where $|T|$ is the area of the triangle T .

So we have $N = 2$, $kPi = 6$; $npPi = 3$; and:

- $P_p = \left\{ \frac{b+c}{2}, \frac{a+c}{2}, \frac{b+a}{2} \right\}$
- $\alpha_0 = -\mathbf{e}_2^0, \alpha_1 = \mathbf{e}_1^0, \alpha_2 = -\mathbf{e}_2^1, \alpha_3 = \mathbf{e}_1^1, \alpha_4 = -\mathbf{e}_2^2, \alpha_5 = \mathbf{e}_1^2$ (effectively, the vector $(-\mathbf{e}_2^m, \mathbf{e}_1^m)$ is orthogonal to the edge $\mathbf{e}^m = (e_1^m, e_2^m)$ with a length equal to the side of the edge or equal to $\int_{e^m} 1$).
- $i_k = \{0, 0, 1, 1, 2, 2\}$,
- $p_k = \{0, 0, 1, 1, 2, 2\}$, $j_k = \{0, 1, 0, 1, 0, 1, 0, 1\}$.

13.2 Which class to add?

Add file `FE_ADD.cpp` in directory `src/femlib` for example first to initialize :

```
#include "error.hpp"
#include "rgraph.hpp"
using namespace std;
#include "RNM.hpp"
#include "fem.hpp"
#include "FESpace.hpp"
#include "AddNewFE.h"
```

```
namespace Fem2D {
```

Then add a class which derive for `public TypeOfFE` like:

```
class TypeOfFE_RTortho : public TypeOfFE { public:
    static int Data[]; // some numbers
    TypeOfFE_RTortho():
        TypeOfFE( 0+3+0, // nb degree of freedom on element
            2, // dimension N of vectorial FE (1 if scalar FE)
            Data, // the array data
            1, // nb of subdivision for plotting
            1, // nb of sub finite element (generally 1)
            6, // number kPi of coef to build the interpolator (13.1)
            3, // number npPi of integration point to build interpolator
            0 // an array to store the coef  $\alpha_k$  to build interpolator
            // here this array is no constant so we have
            // to rebuilt for each element.
        )
    {
        const R2 Pt[] = { R2(0.5,0.5), R2(0.0,0.5), R2(0.5,0.0) };
        // the set of Point in  $\hat{K}$ 

        for (int p=0, kk=0; p<3; p++) {
            P_Pi_h[p]=Pt[p];
            for (int j=0; j<2; j++)
                pij_alpha[kk++] = IPJ(p,p,j); } // definition of  $i_k, p_k, j_k$  in (13.1)

    void FB(const bool * watdd, const Mesh & Th, const Triangle & K,
        const R2 &PHat, RNMK_ & val) const;
```

```
void Pi_h_alpha(const baseFEElement & K,KN_<double> & v) const ;

} ;
```

where the array data is form with the concatenation of five array of size NbDoF and one array of size N.

This array is:

```
int TypeOfFE_RTortho::Data[]={

        //      for each df 0,1,3 :
    3,4,5, //      the support of the node of the df
    0,0,0, //      the number of the df on the node
    0,1,2, //      the node of the df
    0,0,0, //      the df come from which FE (generally 0)
    0,1,2, //      which are de df on sub FE
    0,0 } ; //      for each component j=0,N-1 it give the sub FE associated
```

where the support is a number 0,1,2 for vertex support, 3,4,5 for edge support, and finally 6 for element support.

The function to defined the function ω_i^K , this function return the value of all the basics function or this derivatives in array val, computed at point PHat on the reference triangle corresponding to point R2 $P=K(PHat)$; on the current triangle K.

The index i, j, k of the array $val(i, j, k)$ corresponding to:

i is basic function number on finite element $i \in [0, NoF[$

j is the value of component $j \in [0, N[$

k is the type of computed value $f(P), dx(f)(P), dy(f)(P), \dots i \in [0, last_operatortype[$. Remark for optimization, this value is computed only if $whatd[k]$ is true, and the numbering is defined with

```
enum operatortype { op_id=0,
    op_dx=1, op_dy=2,
    op_dxx=3, op_dyy=4,
    op_dyx=5, op_dxy=5,
    op_dz=6,
    op_dzz=7,
    op_dzx=8, op_dxz=8,
    op_dzy=9, op_dyz=9
};
const int last_operatortype=10;
```

The shape function :

```
void TypeOfFE_RTortho::FB(const bool *whatd,const Mesh & Th,const Triangle & K,
        const R2 & PHat,RNMK_ & val) const
{
    R2 P(K(PHat));
    R2 A(K[0]), B(K[1]), C(K[2]);
    R l0=1-P.x-P.y, l1=P.x, l2=P.y;
    assert(val.N() >=3);
    assert(val.M()==2 );
    val=0;
}
```

```

R a=1./(2*K.area);
R a0=  K.EdgeOrientation(0) * a ;
R a1=  K.EdgeOrientation(1) * a ;
R a2=  K.EdgeOrientation(2) * a ;

                                     // -----
if (whatd[op_id])                    // value of the function
{
    assert (val.K()>op_id);
    RN_ f0(val('.',0,0));           // value first component
    RN_ f1(val('.',1,0));           // value second component
    f1[0] =  (P.x-A.x)*a0;
    f0[0] = -(P.y-A.y)*a0;

    f1[1] =  (P.x-B.x)*a1;
    f0[1] = -(P.y-B.y)*a1;

    f1[2] =  (P.x-C.x)*a2;
    f0[2] = -(P.y-C.y)*a2;
}

                                     // -----
if (whatd[op_dx])                  // value of the dx of function
{
    assert (val.K()>op_dx);
    val(0,1,op_dx) =  a0;
    val(1,1,op_dx) =  a1;
    val(2,1,op_dx) =  a2;
}
if (whatd[op_dy])
{
    assert (val.K()>op_dy);
    val(0,0,op_dy) = -a0;
    val(1,0,op_dy) = -a1;
    val(2,0,op_dy) = -a2;
}

for (int i= op_dy; i< last_operatortype ; i++)
    if (whatd[op_dx])
        assert (op_dy);
}

```

The function to defined the coefficient α_k :

```

void TypeOfFE_RT::Pi_h_alpha(const baseFEelement & K,KN_<double> & v) const
{
    const Triangle & T(K.T);

    for (int i=0,k=0;i<3;i++)
    {
        R2 E(T.Edge(i));
        R signe = T.EdgeOrientation(i) ;
        v[k++]= signe*E.y;
        v[k++] = -signe*E.x;
    }
}

```

Now , we just need to add a new key work in FreeFem++, Two way, with static or dynamic link so at the end of the file, we add :

With dynamic link is very simple (see section C of appendix), just add before the end of FEM2d namespace add:

```
static TypeOfFE_RTortho The_TypeOfFE_RTortho; //
static AddNewFE("RT0Ortho", The_TypeOfFE_RTortho);
} // FEM2d namespace
```

Try with `"/load.link"` command in `examples++-load/` and see `BernardiRaugel.cpp` or `Morley.cpp` new finite element examples.

Otherwise with static link (for expert only), add

```
// let the 2 globals variables
static TypeOfFE_RTortho The_TypeOfFE_RTortho; //
// ----- the name in freefem -----
static ListOfTFE typefemRTortho("RT0Ortho", & The_TypeOfFE_RTortho); //

// link with FreeFem++ do not work with static library .a
// FH so add a extern name to call in init-static-FE
// (see end of FESpace.cpp)
void init_FE_ADD() { };
// --- end ---
} // FEM2d namespace
```

To inforce in loading of this new finite element, we have to add the two new lines close to the end of files `src/femlib/FESpace.cpp` like:

```
// correct Problem of static library link with new make file
void init_static_FE()
{
    extern void init_FE_P2h() ; // list of other FE file.o
    init_FE_P2h() ;
    extern void init_FE_ADD() ; // new line 1
    init_FE_ADD(); // new line 2
}
```

and now you have to change the makefile.

First, create a file `FE_ADD.cpp` contening all this code, like in file `src/femlib/Element_P2h.cpp`, after modifier the `Makefile.am` by adding the name of your file to the variable `EXTRA_DIST` like:

```
# Makefile using Automake + Autoconf
# -----
# $Id$
```

```
# This is not compiled as a separate library because its
# interconnections with other libraries have not been solved.
```

```
EXTRA_DIST=BamgFreeFem.cpp BamgFreeFem.hpp CGNL.hpp CheckPtr.cpp \
ConjuguedGradientNL.cpp DOperator.hpp Drawing.cpp Element_P2h.cpp \
```

```

Element_P3.cpp Element_RT.cpp fem3.hpp fem.cpp fem.hpp FESpace.cpp      \
FESpace.hpp FESpace-v0.cpp FQuadTree.cpp FQuadTree.hpp gibbs.cpp      \
glutdraw.cpp gmres.hpp MatriceCreuse.hpp MatriceCreuse_tpl.hpp        \
MeshPoint.hpp mortar.cpp mshptg.cpp QuadratureFormular.cpp           \
QuadratureFormular.hpp RefCounter.hpp RNM.hpp RNM_opc.hpp RNM_op.hpp   \
RNM_tpl.hpp    FE_ADD.cpp

```

and do in the freefem++ root directory

```

autoreconf
./reconfigure
make

```

For codewarrior compilation add the file in the project and remove the flag in panel PPC linker FreeFem++ Setting Dead-strip Static Initialization Code Flag.

Appendix A

Table of Notations

Here mathematical expressions and corresponding FreeFem++ commands are explained.

A.1 Generalities

δ_{ij} Kronecker delta (0 if $i \neq j$, 1 if $i = j$ for integers i, j)

\forall for all

\exists there exist

i.e. that is

PDE partial differential equation (with boundary conditions)

\emptyset the empty set

\mathbb{N} the set of integers ($a \in \mathbb{N} \Leftrightarrow \text{int } a$); “int” means *long integer* inside FreeFem++

\mathbb{R} the set of real numbers ($a \in \mathbb{R} \Leftrightarrow \text{real } a$); *double* inside FreeFem++

\mathbb{C} the set of complex numbers ($a \in \mathbb{C} \Leftrightarrow \text{complex } a$); *complex;double*

\mathbb{R}^d d -dimensional Euclidean space

A.2 Sets, Mappings, Matrices, Vectors

Let E, F, G be three sets and A subset of E .

$\{x \in E \mid P\}$ the subset of E consisting of the elements possessing the property P

$E \cup F$ the set of elements belonging to E or F

$E \cap F$ the set of elements belonging to E and F

$E \setminus A$ the set $\{x \in E \mid x \notin A\}$

$E + F$ $E \cup F$ with $E \cap F = \emptyset$

$E \times F$ the cartesian product of E and F

E^n the n -th power of E ($E^2 = E \times E$, $E^n = E \times E^{n-1}$)

$f : E \rightarrow F$ the mapping from E into F , i.e., $E \ni x \mapsto f(x) \in F$

I_E **or** I the identity mapping in E , i.e., $I(x) = x \quad \forall x \in E$

$f \circ g$ for $f : F \rightarrow G$ and $g : E \rightarrow F$, $E \ni x \mapsto (f \circ g)(x) = f(g(x)) \in G$ (see Section ??)

$f|_A$ the restriction of $f : E \rightarrow F$ to the subset A of E

$\{a_k\}$ column vector with components a_k

(a_k) row vector with components a_k

$(a_k)^T$ denotes the transpose of a matrix (a_k) , and is $\{a_k\}$

$\{a_{ij}\}$ matrix with components a_{ij} , and $(a_{ij})^T = (a_{ji})$

A.3 Numbers

For two real numbers a, b

$[a, b]$ is the interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$

$]a, b]$ is the interval $\{x \in \mathbb{R} \mid a < x \leq b\}$

$[a, b[$ is the interval $\{x \in \mathbb{R} \mid a \leq x < b\}$

$]a, b[$ is the interval $\{x \in \mathbb{R} \mid a < x < b\}$

A.4 Differential Calculus

$\partial f / \partial x$ the partial derivative of $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with respect to x ($\mathbf{dx}(\mathbf{f})$)

∇f the gradient of $f : \Omega \rightarrow \mathbb{R}$, i.e., $\nabla f = (\partial f / \partial x, \partial f / \partial y)$

div \mathbf{f} **or** $\nabla \cdot \mathbf{f}$ the divergence of $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$, i.e., $\text{div } \mathbf{f} = \partial f_1 / \partial x + \partial f_2 / \partial y$

Δf the Laplacian of $f : \Omega \rightarrow \mathbb{R}$, i.e., $\Delta f = \partial^2 f / \partial x^2 + \partial^2 f / \partial y^2$

A.5 Meshes

Ω usually denotes a domain on which PDE is defined

Γ denotes the boundary of Ω , i.e., $\Gamma = \partial\Omega$ (keyword **border**, see Section 5.1.2)

\mathcal{T}_h the triangulation of Ω , i.e., the set of triangles T_k , where h stands for mesh size (keyword **mesh**, **buildmesh**, see Section 5)

n_t the number of triangles in \mathcal{T}_h (get by `Th.nt`, see “mesh.edp”)

Ω_h denotes the approximated domain $\Omega_h = \cup_{k=1}^{n_t} T_k$ of Ω . If Ω is polygonal domain, then it will be $\Omega = \Omega_h$

Γ_h the boundary of Ω_h

n_v the number of vertices in \mathcal{T}_h (get by `Th.nv`)

$[q^i q^j]$ the segment connecting q^i and q^j

$q^{k_1}, q^{k_2}, q^{k_3}$ the vertices of a triangle T_k with anti-clock direction (get the coordinate of q^{k_j} by `(Th[k-1][j-1].x, Th[k-1][j-1].y)`)

I_Ω the set $\{i \in \mathbb{N} \mid q^i \notin \Gamma_h\}$

A.6 Finite Element Spaces

$L^2(\Omega)$ the set $\left\{ w(x, y) \mid \int_{\Omega} |w(x, y)|^2 dx dy < \infty \right\}$

$$\text{norm: } \|w\|_{0,\Omega} = \left(\int_{\Omega} |w(x, y)|^2 dx dy \right)^{1/2}$$

$$\text{scalar product: } (v, w) = \int_{\Omega} vw$$

$H^1(\Omega)$ the set $\left\{ w \in L^2(\Omega) \mid \int_{\Omega} (|\partial w / \partial x|^2 + |\partial w / \partial y|^2) dx dy < \infty \right\}$

$$\text{norm: } \|w\|_{1,\Omega} = (\|w\|_{0,\Omega}^2 + \|\nabla u\|_{0,\Omega}^2)^{1/2}$$

$H^m(\Omega)$ the set $\left\{ w \in L^2(\Omega) \mid \int_{\Omega} \frac{\partial^{|\alpha|} w}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \in L^2(\Omega) \quad \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, |\alpha| = \alpha_1 + \alpha_2 \right\}$

$$\text{scalar product: } (v, w)_{1,\Omega} = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha v D^\alpha w$$

$H_0^1(\Omega)$ the set $\{w \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma\}$

$L^2(\Omega)^2$ denotes $L^2(\Omega) \times L^2(\Omega)$, and also $H^1(\Omega)^2 = H^1(\Omega) \times H^1(\Omega)$

V_h denotes the finite element space created by “ **fespace** Vh(Th,*)” in FreeFem++ (see Section 6 for “*”)

$\Pi_h f$ the projection of the function f into V_h (“ **func** f=x^2*y^3; Vh v = f;” means $v = \Pi_h f$)

$\{v\}$ for FE-function v in V_h means the column vector $(v_1, \dots, v_M)^T$ if $v = v_1\phi_1 + \dots + v_M\phi_M$, which is shown by “ **fespace** Vh(Th,P2); Vh v; cout << v[] << endl;”

Appendix B

Grammar

B.1 The bison grammar

```
start:  input ENDOFFILE;

input:  instructions ;

instructions: instruction
          | instructions instruction ;

list_of_id_args:
    | id
    | id '=' no_comma_expr
    | FESPACE id
    | type_of_dcl id
    | type_of_dcl '&' id
    | '[' list_of_id_args ']'
    | list_of_id_args ',' id
    | list_of_id_args ',' '[' list_of_id_args ']'
    | list_of_id_args ',' id '=' no_comma_expr
    | list_of_id_args ',' FESPACE id
    | list_of_id_args ',' type_of_dcl id
    | list_of_id_args ',' type_of_dcl '&' id ;

list_of_id1: id
            | list_of_id1 ',' id ;

id: ID | FESPACE ;

list_of_dcls:  ID
              | ID '=' no_comma_expr
              | ID '(' parameters_list ')'
              | list_of_dcls ',' list_of_dcls ;

parameters_list:
    no_set_expr
    | FESPACE ID
    | ID '=' no_set_expr
```

```

    | parameters_list ',' no_set_expr
    | parameters_list ',' id '=' no_set_expr ;

type_of_dcl:    TYPE
               | TYPE '[' TYPE ']' ;

ID_space:
    ID
    | ID '[' no_set_expr ']'
    | ID '=' no_set_expr
    | '[' list_of_id1 ']'
    | '[' list_of_id1 ']' '[' no_set_expr ']'
    | '[' list_of_id1 ']' '=' no_set_expr ;

ID_array_space:
    ID '(' no_set_expr ')'
    | '[' list_of_id1 ']' '(' no_set_expr ')' ;

fespace: FESPACE ;

spaceIDa  :      ID_array_space
           |      spaceIDa ',' ID_array_space  ;

spaceIDb  :      ID_space
           |      spaceIDb ',' ID_space  ;

spaceIDs  :      fespace                      spaceIDb
           |      fespace '[' TYPE ']'      spaceIDa      ;

fespace_def: ID '(' parameters_list ')' ;

fespace_def_list: fespace_def
                 | fespace_def_list ',' fespace_def ;

declaration:  type_of_dcl list_of_dcls ';'
              | 'fespace' fespace_def_list      ';'
              | spaceIDs ';'
              | FUNCTION ID '=' Expr ';'
              | FUNCTION type_of_dcl ID '(' list_of_id_args ')' '{' instructions'}'
              | FUNCTION ID '(' list_of_id_args ')' '=' no_comma_expr ';' ;

begin: '{' ;
end:   '}' ;

for_loop:  'for' ;
while_loop: 'while' ;

instruction:  ';'
            | 'include' STRING
            | 'load'  STRING
            | Expr ';'
            | declaration
            | for_loop '(' Expr ';' Expr ';' Expr ')' instruction
            | while_loop '(' Expr ')' instruction
            | 'if' '(' Expr ')' instruction

```

```

| 'if' '(' Expr ')' instruction ELSE instruction
| begin instructions end
| 'border' ID border_expr
| 'border' ID '[' array ']' ';'
| 'break' ';'
| 'continue' ';'
| 'return' Expr ';' ;

```

```
bornes: '(' ID '=' Expr ',' Expr ')' ;
```

```
border_expr: bornes instruction ;
```

```
Expr: no_comma_expr
      | Expr ',' Expr ;
```

```
unop: '-'
      | '+'
      | '!'
      | '++'
      | '--' ;
```

```
no_comma_expr:
  no_set_expr
| no_set_expr '=' no_comma_expr
| no_set_expr '+=' no_comma_expr
| no_set_expr '-=' no_comma_expr
| no_set_expr '*=' no_comma_expr
| no_set_expr '/=' no_comma_expr
| no_set_expr '.*=' no_comma_expr
| no_set_expr './=' no_comma_expr ;
```

```
no_set_expr:
  no_ternary_expr
| no_ternary_expr '?' no_set_expr ':' no_set_expr ;
```

```
no_ternary_expr:
  unary_expr
| no_ternary_expr '*' no_ternary_expr
| no_ternary_expr '.*' no_ternary_expr
| no_ternary_expr './' no_ternary_expr
| no_ternary_expr '/' no_ternary_expr
| no_ternary_expr '%' no_ternary_expr
| no_ternary_expr '+' no_ternary_expr
| no_ternary_expr '-' no_ternary_expr
| no_ternary_expr '<<' no_ternary_expr
| no_ternary_expr '>>' no_ternary_expr
| no_ternary_expr '&' no_ternary_expr
| no_ternary_expr '&&' no_ternary_expr
| no_ternary_expr '|' no_ternary_expr
| no_ternary_expr '||' no_ternary_expr
| no_ternary_expr '<' no_ternary_expr
| no_ternary_expr '<=' no_ternary_expr
```

```

    | no_ternary_expr '>' no_ternary_expr
    | no_ternary_expr '>=' no_ternary_expr
    | no_ternary_expr '==' no_ternary_expr
    | no_ternary_expr '!=' no_ternary_expr ;

sub_script_expr:
    no_set_expr
|   ':'
|   no_set_expr ':' no_set_expr
|   no_set_expr ':' no_set_expr ':' no_set_expr ;

parameters:
    |   no_set_expr
    |   FESPACE
    |   id '=' no_set_expr
    |   sub_script_expr
    |   parameters ',' FESPACE
    |   parameters ',' no_set_expr
    |   parameters ',' id '=' no_set_expr ;

array:   no_comma_expr
    |   array ',' no_comma_expr ;

unary_expr:
    pow_expr
    |   unop pow_expr %prec UNARY ;

pow_expr: primary
    |   primary '^' unary_expr
    |   primary '_' unary_expr
    |   primary '/' ;                                     // transpose

primary:
    ID
    |   LNUM
    |   DNUM
    |   CNUM
    |   STRING
    |   primary '(' parameters ')'
    |   primary '[' Expr ']'
    |   primary '[' ']'
    |   primary '.' ID
    |   primary '++'
    |   primary '--'
    |   TYPE '(' Expr ')' ;
    |   '(' Expr ')'
    |   '[' array ']' ;

```

B.2 The Types of the languages, and cast

B.3 All the operators

```

- CG,   type :<TypeSolveMat>
- Cholesky,   type :<TypeSolveMat>
- Crout,   type :<TypeSolveMat>
- GMRES,   type :<TypeSolveMat>
- LU,   type :<TypeSolveMat>
- LinearCG,   type :<Polymorphic>   operator() :
  (   <long> :   <Polymorphic>, <KN<double> *>, <KN<double> *> )

- N,   type :<Fem2D::R3>
- NoUseOfWait,   type :<bool *>
- P,   type :<Fem2D::R3>
- P0,   type :<Fem2D::TypeOfFE>
- P1,   type :<Fem2D::TypeOfFE>
- P1nc,   type :<Fem2D::TypeOfFE>
- P2,   type :<Fem2D::TypeOfFE>
- RT0,   type :<Fem2D::TypeOfFE>
- RTmodif,   type :<Fem2D::TypeOfFE>
- abs,   type :<Polymorphic>   operator() :
  (   <double> :   <double> )

- acos,   type :<Polymorphic>   operator() :
  (   <double> :   <double> )

- acosh,   type :<Polymorphic>   operator() :
  (   <double> :   <double> )

- adaptmesh,   type :<Polymorphic>   operator() :
  (   <Fem2D::Mesh> :   <Fem2D::Mesh>... )

- append,   type :<std::ios_base::openmode>
- asin,   type :<Polymorphic>   operator() :
  (   <double> :   <double> )

- asinh,   type :<Polymorphic>   operator() :
  (   <double> :   <double> )

- atan,   type :<Polymorphic>   operator() :
  (   <double> :   <double> )
  (   <double> :   <double>, <double> )

- atan2,   type :<Polymorphic>   operator() :
  (   <double> :   <double>, <double> )

- atanh,   type :<Polymorphic>   operator() :
  (   <double> :   <double> )

```

```

- buildmesh, type :<Polymorphic> operator() :
  (    <Fem2D::Mesh> :    <E_BorderN> )

- buildmeshborder, type :<Polymorphic> operator() :
  (    <Fem2D::Mesh> :    <E_BorderN> )

- cin, type :<istream>
- clock, type :<Polymorphic>
  (    <double> :    )

- conj, type :<Polymorphic> operator() :
  (    <complex> :    <complex> )

- convect, type :<Polymorphic> operator() :
  (    <double> :    <E_Array>, <double>, <double> )

- cos, type :<Polymorphic> operator() :
  (    <double> :    <double> )
  (    <complex> :    <complex> )

- cosh, type :<Polymorphic> operator() :
  (    <double> :    <double> )
  (    <complex> :    <complex> )

- cout, type :<ostream>
- dumptable, type :<Polymorphic> operator() :
  (    <ostream> :    <ostream> )

- dx, type :<Polymorphic> operator() :
  (    <LinearComb<MDroit, C_F0>> :    <LinearComb<MDroit, C_F0>> )
  (    <double> :    <std::pair<FEbase<double> *, int>> )
  (    <LinearComb<MGauche, C_F0>> :    <LinearComb<MGauche, C_F0>> )

- dy, type :<Polymorphic> operator() :
  (    <LinearComb<MDroit, C_F0>> :    <LinearComb<MDroit, C_F0>> )
  (    <double> :    <std::pair<FEbase<double> *, int>> )
  (    <LinearComb<MGauche, C_F0>> :    <LinearComb<MGauche, C_F0>> )

- endl, type :<char>
- exec, type :<Polymorphic> operator() :
  (    <long> :    <string> )

- exit, type :<Polymorphic> operator() :
  (    <long> :    <long> )

- exp, type :<Polymorphic> operator() :
  (    <double> :    <double> )
  (    <complex> :    <complex> )

```



```

- false, type :<bool>
- imag, type :<Polymorphic> operator() :
  ( <double> : <complex> )

- int1d, type :<Polymorphic> operator() :
  ( <CDomainOfIntegration> : <Fem2D::Mesh>... )

- int2d, type :<Polymorphic> operator() :
  ( <CDomainOfIntegration> : <Fem2D::Mesh>... )

- intalldges, type :<Polymorphic>
operator( :
  ( <CDomainOfIntegration> : <Fem2D::Mesh>... )

- jump, type :<Polymorphic>
operator( :
  ( <LinearComb<MDroit, C_F0>> : <LinearComb<MDroit, C_F0>> )
  ( <double> : <double> )
  ( <complex> : <complex> )
  ( <LinearComb<MGauche, C_F0>> : <LinearComb<MGauche, C_F0>> )

- label, type :<long *>
- log, type :<Polymorphic> operator() :
  ( <double> : <double> )
  ( <complex> : <complex> )

- log10, type :<Polymorphic> operator() :
  ( <double> : <double> )

- max, type :<Polymorphic> operator() :
  ( <double> : <double>, <double> )
  ( <long> : <long>, <long> )

- mean, type :<Polymorphic>
operator( :
  ( <double> : <double> )
  ( <complex> : <complex> )

- min, type :<Polymorphic> operator() :
  ( <double> : <double>, <double> )
  ( <long> : <long>, <long> )

- movemesh, type :<Polymorphic> operator() :
  ( <Fem2D::Mesh> : <Fem2D::Mesh>, <E_Array>... )

- norm, type :<Polymorphic>
operator( :
  ( <double> : <std::complex<double>> )

- nuTriangle, type :<long>

```

```

- nuEdge, type :<long>
- on, type :<Polymorphic> operator() :
  ( <BC_set<double>> : <long>... )

- otherside, type :<Polymorphic>
operator( :
  ( <LinearComb<MDroit, C_F0>> : <LinearComb<MDroit, C_F0>> )
  ( <LinearComb<MGauche, C_F0>> : <LinearComb<MGauche, C_F0>> )

- pi, type :<double>
- plot, type :<Polymorphic> operator() :
  ( <long> : ... )

- pow, type :<Polymorphic> operator() :
  ( <double> : <double>, <double> )
  ( <complex> : <complex>, <complex> )

- qf1pE, type :<Fem2D::QuadratureFormular1d>
- qf1pT, type :<Fem2D::QuadratureFormular>
- qf1pTlump, type :<Fem2D::QuadratureFormular>
- qf2pE, type :<Fem2D::QuadratureFormular1d>
- qf2pT, type :<Fem2D::QuadratureFormular>
- qf2pT4P1, type :<Fem2D::QuadratureFormular>
- qf3pE, type :<Fem2D::QuadratureFormular1d>
- qf5pT, type :<Fem2D::QuadratureFormular>

- readmesh, type :<Polymorphic> operator() :
  ( <Fem2D::Mesh> : <string> )

- real, type :<Polymorphic> operator() :
  ( <double> : <complex> )

- region, type :<long *>
- savemesh, type :<Polymorphic> operator() :
  ( <Fem2D::Mesh> : <Fem2D::Mesh>, <string>... )

- sin, type :<Polymorphic> operator() :
  ( <double> : <double> )
  ( <complex> : <complex> )

- sinh, type :<Polymorphic> operator() :
  ( <double> : <double> )
  ( <complex> : <complex> )

- sqrt, type :<Polymorphic> operator() :
  ( <double> : <double> )
  ( <complex> : <complex> )

- square, type :<Polymorphic> operator() :
  ( <Fem2D::Mesh> : <long>, <long> )

```

```
(    <Fem2D::Mesh> :    <long>, <long>, <E_Array> )

- tan,  type :<Polymorphic>  operator() :
(    <double> :    <double> )

- true,  type :<bool>
- trunc,  type :<Polymorphic>  operator() :
(    <Fem2D::Mesh> :    <Fem2D::Mesh>, <bool> )

- verbosity,  type :<long *>
- wait,  type :<bool *>
- x,  type :<double *>
- y,  type :<double *>
- z,  type :<double *>
```


Appendix C

Dynamical link

Now, it's possible to add built-in fonctionnalites in FreeFem++ under the three environments Linux, Windows and MacOS X 10.3 or newer. It is a good idea to, first try the example `load.edp` in directory `example++-load`.

You will need to install a c++ compiler (generally g++/gcc compiler) to compile your function.

Windows Install the cygwin environment or the mingw

MacOs Install the developer tools xcode on the apple DVD

Linux/Unix Install the correct compiler (gcc for instance)

Now, assume that you are in a shell window (a cygwin window under Windows) in the directory `example++-load`. Remark that in the sub directory `include` they are all the FreeFem++ include file to make the link with FreeFem++ .

Note C.1 *If you try to load dynamically a file with command `load "xxx"`*

- *Under unix (Linux or MacOS), the file `xxx.so` will be loaded so it must be either in the search directory of routine `dlopen` (see the environment variable `$LD_LIBRARY_PATH` or in the current directory, and the suffix `".so"` or the prefix `"/"` is automatically added.*
- *Under Windows, The file `xxx.dll` will be loaded so it must be in the `loadLibrary` search directory which includes the directory of the application,*

The compilation of your module: the script `ff-c++` compiles and makes the link with FreeFem++ , but be careful, the script has no way to know if you try to compile for a pure Windows environment or for a cygwin environment so to build the load module under cygwin you must add the `-cygwin` parameter.

C.1 A first example `myfunction.cpp`

The following defines a new function call `myfunction` with no parameter, but using the x, y current value.

```

#include <iostream>
#include <cfloat>
using namespace std;
#include "error.hpp"
#include "AFunction.hpp"
#include "rgraph.hpp"
#include "RNM.hpp"
#include "fem.hpp"
#include "FESpace.hpp"
#include "MeshPoint.hpp"

using namespace Fem2D;
double myfunction(Stack stack)
{
    // to get FreeFem++ data
    MeshPoint &mp= *MeshPointStack(stack); // the struct to get x,y, normal ,
    value
    double x= mp.P.x; // get the current x value
    double y= mp.P.y; // get the current y value
    // cout << "x = " << x << " y=" << y << endl;

    return sin(x)*cos(y);
}

```

Now the Problem is to build the link with FreeFem++ , to do that we need two classes, one to call the function myfunction

All FreeFem++ evaluable expression must be a struct/class C++ which derive from E_F0. By default this expression does not depend of the mesh position, but if they derive from E_F0mps the expression depends of the mesh position, and for more details see [12].

```

// A class build the link with FreeFem++
// generally this class are already in AFunction.hpp
// but unfortunately, I have no simple function with no parameter
// in FreeFem++ depending of the mesh,

template<class R>
class OneOperator0s : public OneOperator {

    // the class to defined a evaluated a new function
    // It must devive from E_F0 if it is mesh independent
    // or from E_F0mps if it is mesh dependent

    class E_F0_F :public E_F0mps { public:
        typedef R (*func)(Stack stack) ;
        func f; // the pointeur to the fnction myfunction
        E_F0_F(func ff) : f(ff) {}

        // the operator evaluation in FreeFem++
        AnyType operator()(Stack stack) const {return SetAny<R>( f(stack)) ;}

    };

    typedef R (*func)(Stack ) ;
    func f;
public:
    // the function which build the FreeFem++ byte code
    E_F0 * code(const basicAC_F0 & ) const { return new E_F0_F(f); }
    // the constructor to say ff is a function without parameter

```

```

// and returning a R
OneOperator0s(func ff): OneOperator(map_type[typeid(R).name()], f(ff) {}
};

```

To finish we must add this new function in FreeFem++ table , to do that include :

```

void init() {
    Global.Add("myfunction", "(", new OneOperator0s<double>(myfunction));
}
LOADFUNC(init);

```

It will be called automatically at load module time.

To compile and link, use the ff-c++ script :

```

% ff-c++ myfunction.cpp
g++ -c -g -Iinclude myfunction.cpp
g++ -bundle -undefined dynamic_lookup -g myfunction.o -o ./myfunction.dylib

```

To, try the simple example under Linux or MacOS, do

```

% FreeFem++-nw load.edp
-- FreeFem++ v 1.4800028 (date Tue Oct  4 11:56:46 CEST 2005)
file : load.edp
Load: lg_fem lg_mesh eigenvalue  UMPACK
1 :                               // Example of dynamic function load
2 :                               // -----
3 :                               // Id: freefem + doc.tex, v1.1102010/06/0411:27:24hechtExp
4 :
5 :  load "myfunction"

load: myfunction
load: dlopen(./myfunction) = 0xb01cc0

6 :  mesh Th=square(5,5);
7 :  fespace Vh(Th,P1);
8 :  Vh uh=myfunction(); // warning do not forget ()
9 :  cout << uh[].min << " " << uh[].max << endl;
10 :  sizestack + 1024 =1240  ( 216 )

-- square mesh : nb vertices  =36 ,  nb triangles = 50 ,  nb boundary edges 20
Nb of edges on Mortars  = 0
Nb of edges on Boundary = 20, neb = 20
Nb Of Nodes = 36
Nb of DF = 36
0 0.841471
times: compile 0.05s, execution -3.46945e-18s
CodeAlloc : nb ptr 1394, size :71524
Bien: On a fini Normalement

```

Under Windows, launch FreeFem++ with the mouse (or ctrl O) on the example.

C.2 Example: Discrete Fast Fourier Transform

This will add FFT to FreeFem++ , taken from <http://www.fftw.org/>. To download and install under download/include just go in download/fftw and trymake.

The 1D dfft (fast discret fourier transform) for a simple array f of size n is defined by the following formula

$$\text{dfft}(f, \varepsilon)_k = \sum_{j=0}^{n-1} f_j e^{\varepsilon 2\pi i k j / n}$$

The 2D DFFT for an array of size $N = n \times m$ is

$$\text{dfft}(f, m, \varepsilon)_{k+nl} = \sum_{j'=0}^{m-1} \sum_{j=0}^{n-1} f_{i+nj} e^{\varepsilon 2\pi i (kj/n + lj'/m)}$$

Remark: the value n is given by $\text{size}(f)/m$, and the numbering is row-major order.

So the classical discrete DFFT is $\hat{f} = \text{dfft}(f, -1)/\sqrt{n}$ and the reverse dFFT $f = \text{dfft}(\hat{f}, 1)/\sqrt{n}$

Remark: the 2D Laplace operator is

$$f(x, y) = 1/\sqrt{N} \sum_{j'=0}^{m-1} \sum_{j=0}^{n-1} \hat{f}_{i+nj} e^{\varepsilon 2\pi i (xj + yj')}$$

and we have

$$f_{k+nl} = f(k/n, l/m)$$

So

$$\widehat{\Delta f_{kl}} = -((2\pi)^2((\tilde{k})^2 + (\tilde{l})^2))\widehat{f_{kl}}$$

where $\tilde{k} = k$ if $k \leq n/2$ else $\tilde{k} = k - n$ and $\tilde{l} = l$ if $l \leq m/2$ else $\tilde{l} = l - m$.

And to have a real function we need all modes to be symmetric around zero, so n and m must be odd.

Compile to build a new library

```
% ff-c++ dfft.cpp ../download/install/lib/libfftw3.a -I../download/install/include
export MACOSX_DEPLOYMENT_TARGET=10.3
g++ -c -Iinclude -I../download/install/include dfft.cpp
g++ -bundle -undefined dynamic_lookup dfft.o -o ../dfft.dylib ../download/install/lib/libfftw3.a
```

To test ,

```
-- FreeFem++ v 1.4800028 (date Mon Oct 10 16:53:28 EEST 2005)
file : dfft.edp
Load: lg_fem cadna lg_mesh eigenvalue UMFPACK
1 : // Example of dynamic function load
2 : // -----
3 : // Id: freefem++doc.tex,v1.1102010/06/0411:27:24hechtExp
4 : // Discret Fast Fourier Transform
5 : // -----
6 : load "dfft" load: init dfft

load: dlopen(dfft.dylib) = 0x2b0c700
```



```

7 :
8 : int nx=32,ny=16,N=nx*ny;
9 : //    warning the Fourier space is not exactly the unite square due to
periodic conditions
10 : mesh Th=square(nx-1,ny-1,[(nx-1)*x/nx,(ny-1)*y/ny]);
11 :           //    warring the numbering is of the vertices (x,y) is
12 :           //    given by  $i = x/nx + nx * y/ny$ 
13 :
14 : fespace Vh(Th,P1);
15 :
16 : func f1 = cos(2*x*2*pi)*cos(3*y*2*pi);
17 : Vh<complex> u=f1,v;
18 : Vh w=f1;
19 :
20 :
21 : Vh ur,ui;
22 : //    in dfft the matrix n,m is in row-major order ann array n,m is
23 : //    store j + m* i ( the transpose of the square numbering )
24 : v[]=dfft(u[],ny,-1);
25 : u[]=dfft(v[],ny,+1);
26 : u[] /= complex(N);
27 : v = f1-u;
28 : cout << " diff = "<< v[].max << " " << v[].min << endl;
29 : assert( norm(v[].max) < 1e-10 && norm(v[].min) < 1e-10 ) ;
30 : //    ----- a more hard example -----
31 : //    Lapacien en FFT
32 : //     $-\Delta u = f$  with biperiodic condition
33 : func f = cos(3*2*pi*x)*cos(2*2*pi*y); //
34 : func ue = +(1./(square(2*pi)*13.))*cos(3*2*pi*x)*cos(2*2*pi*y); //

35 : Vh<complex> ff = f;
36 : Vh<complex> fhat;
37 : fhat[] = dfft(ff[],ny,-1);
38 :
39 : Vh<complex> wij;
40 : //    warning in fact we take mode between -nx/2, nx/2 and -ny/2,ny/2
41 : //    thank to the operator ? :
42 : wij = square(2.*pi)*(square(( x<0.5?x*nx:(x-1)*nx))
+ square((y<0.5?y*ny:(y-1)*ny)));
43 : wij[][0] = 1e-5; //    to remove div / 0
44 : fhat[] = fhat[]./ wij[]; //
45 : u[]=dfft(fhat[],ny,1);
46 : u[] /= complex(N);
47 : ur = real(u); //    the solution
48 : w = real(ue); //    the exact solution
49 : plot(w,ur,value=1 ,cmm=" ue ", wait=1);
50 : w[] -= ur[]; //    array sub
51 : real err= abs(w[].max)+abs(w[].min) ;
52 : cout << " err = " << err << endl;
53 : assert( err < 1e-6);
54 : sizestack + 1024 =3544 ( 2520 )

-----CheckPtr:-----init execution ----- NbUndelPtr 2815 Alloc: 111320 NbPtr
6368
-- square mesh : nb vertices =512 , nb triangles = 930 , nb boundary edges 92

```

```

Nb of edges on Mortars = 0
Nb of edges on Boundary = 92, neb = 92
Nb Of Nodes = 512
Nb of DF = 512
0x2d383d8 -1 16 512 n: 16 m:32
dfft 0x402bc08 = 0x4028208 n = 16 32 sign = -1
--- --- ---0x2d3ae08 1 16 512 n: 16 m:32
dfft 0x4028208 = 0x402bc08 n = 16 32 sign = 1
--- --- --- diff = (8.88178e-16,3.5651e-16) (-6.66134e-16,-3.38216e-16)
0x2d3cfeb8 -1 16 512 n: 16 m:32
dfft 0x402de08 = 0x402bc08 n = 16 32 sign = -1
--- --- ---0x2d37ff8 1 16 512 n: 16 m:32
dfft 0x4028208 = 0x402de08 n = 16 32 sign = 1
--- --- --- err = 3.6104e-12
times: compile 0.13s, execution 2.05s
-----CheckPtr:-----end execution -- ----- NbUndelPtr 2815 Alloc: 111320
NbPtr 26950
CodeAlloc : nb ptr 1693, size :76084
Bien: On a fini Normalement
CheckPtr:Nb of undelete pointer is 2748 last 114
CheckPtr:Max Memory used 228.531 kbytes Memory undelete 105020

```

C.3 Load Module for Dervieux' P0-P1 Finite Volume Method

the associated edp file is examples++-load/convect_dervieux.edp

```

// ----- Implementation of P1-P0 FVM-FEM -----
// ----- Id: freefem++doc.tex,v1.1102010/06/0411:27:24hechtExp -----
// compile and link with ff-c++ mat_dervieux.cpp (i.e. the file name
without .cpp)
#include <iostream>
#include <cfloat>
#include <cmath>
using namespace std;
#include "error.hpp"
#include "AFunction.hpp"
#include "rgraph.hpp"
#include "RNM.hpp"

// remove problem of include
#undef HAVE_LIBUMFPACK
#undef HAVE_CADNA
#include "MatriceCreuse_tpl.hpp"
#include "MeshPoint.hpp"
#include "lgfem.hpp"
#include "lgsolver.hpp"
#include "problem.hpp"

class MatrixUpWind0 : public E_F0mps { public:
    typedef Matrice_Creuse<R> * Result;
    Expression emat,expTh,expc,expul,expu2;
    MatrixUpWind0(const basicAC_F0 & args)
    {

```

```

    args.SetNameParam();
    emat =args[0];
    expTh= to<pmesh>(args[1]);
    expc = CastTo<double>(args[2]);
    // the matrix expression
    // a the expression to get the mesh
    // the expression to get c (must be a
double)
    // a array expression [ a, b]
    const E_Array * a= dynamic_cast<const E_Array*>((Expression) args[3]);
    if (a->size() != 2) CompileError("syntax: MatrixUpWind0(Th,rhi,[u1,u2])");
    int err =0;
    expu1= CastTo<double>((*a)[0]); // fist exp of the array (must be a
double)
    expu2= CastTo<double>((*a)[1]); // second exp of the array (must be a
double)
}

~MatrixUpWind0(){
}

static ArrayOfaType typeargs()
{ return ArrayOfaType(atype<Matrice_Creuse<R>*>(),
    atype<pmesh>(),atype<double>(),atype<E_Array>());}
static E_F0 * f(const basicAC_F0 & args){ return new MatrixUpWind0(args);}

AnyType operator()(Stack s) const ;

};

int fvmP1P0(double q[3][2], double u[2],double c[3], double a[3][3], double where[3]
)
{
    // computes matrix a on a triangle for the
Dervieux FVM
    for(int i=0;i<3;i++) for(int j=0;j<3;j++) a[i][j]=0;

    for(int i=0;i<3;i++){
        int ip = (i+1)%3, ipp =(ip+1)%3;
        double unL =-((q[ip][1]+q[i][1]-2*q[ipp][1])*u[0]
            -(q[ip][0]+q[i][0]-2*q[ipp][0])*u[1])/6;
        if(unL>0) { a[i][i] += unL; a[ip][i]-=unL;}
        else{ a[i][ip] += unL; a[ip][ip]-=unL;}
        if(where[i]&&where[ip]){ // this is a boundary edge
            unL=((q[ip][1]-q[i][1])*u[0] -(q[ip][0]-q[i][0])*u[1])/2;
            if(unL>0) { a[i][i]+=unL; a[ip][ip]+=unL;}
        }
    }
    return 1;
}

// the evaluation routine
AnyType MatrixUpWind0::operator()(Stack stack) const
{
    Matrice_Creuse<R> * sparse_mat =GetAny<Matrice_Creuse<R>*> ((*emat)(stack));
    MatriceMorse<R> * amorse =0;
    MeshPoint *mp(MeshPointStack(stack)) , mps=*mp;
    Mesh * pTh = GetAny<pmesh> ((*expTh)(stack));
    ffassert(pTh);
    Mesh & Th (*pTh);

```

```

{
  map< pair<int,int>, R> Aij;
  KN<double> cc(Th.nv);
  double infini=DBL_MAX;
  cc=infini;
  for (int it=0;it<Th.nt;it++)
    for (int iv=0;iv<3;iv++)
    {
      int i=Th(it,iv);
      if ( cc[i]==infini) { // if nuset the set
        mp->setP(&Th,it,iv);
        cc[i]=GetAny<double>((*expc)(stack));
      }
    }

  for (int k=0;k<Th.nt;k++)
  {
    const Triangle & K(Th[k]);
    const Vertex & A(K[0]), &B(K[1]),&C(K[2]);
    R2 Pt(1./3.,1./3.);
    R u[2];
    MeshPointStack(stack)->set(Th,K(Pt),Pt,K,K.lab);
    u[0] = GetAny< R> ( (*expu1)(stack) ) ;
    u[1] = GetAny< R> ( (*expu2)(stack) ) ;

    int ii[3] = { Th(A), Th(B),Th(C) };
    double q[3][2]= { { A.x,A.y } , {B.x,B.y} , {C.x,C.y} } ; // coordinates
of 3 vertices (input)
    double c[3]={cc[ii[0]],cc[ii[1]],cc[ii[2]]};
    double a[3][3], where[3]={A.lab,B.lab,C.lab};
    if (fvmP1P0(q,u,c,a,where) )
    {
      for (int i=0;i<3;i++)
        for (int j=0;j<3;j++)
          if (fabs(a[i][j]) >= 1e-30)
            { Aij[make_pair(ii[i],ii[j])]+=a[i][j];
            }
    }
  }

  amorse= new MatriceMorse<R>(Th.nv,Th.nv,Aij,false);
}
sparse_mat->pUh=0;
sparse_mat->pVh=0;
sparse_mat->A.master(amorse);
sparse_mat->typemat=(amorse->n == amorse->m) ? TypeSolveMat(TypeSolveMat::GMRES)
: TypeSolveMat(TypeSolveMat::NONESQUARE); // none square matrice (morse)
*mp=mps;

if(verbosity>3) { cout << " End Build MatrixUpWind : " << endl;}

return sparse_mat;
}

void init()
{
  cout << " load: init Mat Chacon " << endl;

```

```

        Global.Add("MatUpWind0", "(" , new OneOperatorCode<MatrixUpWind0 >( ));
    }

LOADFUNC(init);

```

C.4 More on Adding a new finite element

First read the section 13 of the appendix, we add two new finite elements examples in the directory `examples++-load`.

The Bernardi-Raugel Element The Bernardi-Raugel finite element is meant to solve the Navier Stokes equations in u, p formulation; the velocity space P_K^{br} is minimal to prove the inf-sup condition with piecewise constant pressure by triangle. The finite element space V_h is

$$V_h = \{u \in H^1(\Omega)^2; \quad \forall K \in T_h, u|_K \in P_K^{br}\}$$

where

$$P_K^{br} = \text{span}\{\lambda_i^K e_k\}_{i=1,2,3,k=1,2} \cup \{\lambda_i^K \lambda_{i+1}^K n_{i+2}^K\}_{i=1,2,3}$$

with notation $4 = 1, 5 = 2$ and where λ_i^K are the barycentric coordinates of the triangle K , $(e_k)_{k=1,2}$ the canonical basis of \mathbb{R}^2 and n_k^K the outer normal of triangle K opposite to vertex k .

```

//      The P2BR finite element :  the Bernadi Raugel Finite Element
//      F. Hecht, decembre 2005
//      -----
//      See Bernardi, C., Raugel, G.: Analysis of some finite elements for the
Stokes problem. Math. Comp. 44, 71-79 (1985).
//      It is a 2d coupled FE
//      the Polynomial space is  $P_1^2 + 3$  normals bubbles edges function ( $P_2$ )
//      the degree of freedom is 6 values at of the 2 componants at the 3
vertices
//      and the 3 flux on the 3 edges
//      So 9 degrees of freedom and N= 2.

//      ----- related files:
//      to check and validate : testFE.edp
//      to get a real example : NSP2BRP0.edp
//      -----

//      -----

#include "error.hpp"
#include "AFunction.hpp"
#include "rgraph.hpp"
using namespace std;
#include "RNM.hpp"
#include "fem.hpp"
#include "FESpace.hpp"
#include "AddNewFE.h"

```

```

namespace Fem2D {

class TypeOfFE_P2BRLagrange : public TypeOfFE { public:
    static int Data[];

    TypeOfFE_P2BRLagrange(): TypeOfFE(6+3+0,
        2,
        Data,
        4,
        1,
        6+3*(2+2),          // nb coef to build interpolation
        9,                  // np point to build interpolation
        0)
    {
        ....                // to long see the source
    }
    void FB(const bool * whatd, const Mesh & Th,const Triangle & K,const R2 &P,
    RNMK_ & val) const;
    void TypeOfFE_P2BRLagrange::Pi_h_alpha(const baseFEelement & K,KN_<double> &
    v) const;
    } ;

    // on what nu df on node node of df

int TypeOfFE_P2BRLagrange::Data[]={
    0,0, 1,1, 2,2, 3,4,5,
    0,1, 0,1, 0,1, 0,0,0,
    0,0, 1,1, 2,2, 3,4,5,
    0,0, 0,0, 0,0, 0,0,0,
    0,1, 2,3, 4,5, 6,7,8,
    0,0
};

void TypeOfFE_P2BRLagrange::Pi_h_alpha(const baseFEelement & K,KN_<double> & v) const
{
    const Triangle & T(K.T);
    int k=0;
    // coef pour les 3 sommets fois le 2 composantes
    for (int i=0;i<6;i++)
        v[k++]=1;
    // integration sur les aretes
    for (int i=0;i<3;i++)
    {
        R2 N(T.Edge(i).perp());
N    *= T.EdgeOrientation(i)*0.5 ;
        v[k++]= N.x;
        v[k++]= N.y;
        v[k++]= N.x;
        v[k++]= N.y;
    }
}

void TypeOfFE_P2BRLagrange::FB(const bool * whatd,const Mesh & ,const Triangle
& K,const R2 & P,RNMK_ & val) const
{
    ....                // to long see the source
}

```

```

//      ---- cooking to add the finite elemet to freefem table -----
//      a static variable to def the finite element
static TypeOfFE_P2BRLagrange P2LagrangeP2BR;
//      now adding FE in FreeFem++ table
static AddNewFE P2BR("P2BR",&P2LagrangeP2BR);
//      --- end cooking
//      end FEM2d namespace
}

```

A way to check the finite element

```

load "BernadiRaugel"
//      a macro the compute numerical derivative
macro DD(f,hx,hy) ( (f(x1+hx,y1+hy)-f(x1-hx,y1-hy))/(2*(hx+hy))) //
mesh Th=square(1,1,[10*(x+y/3),10*(y-x/3)]);

real x1=0.7,y1=0.9, h=1e-7;
int it1=Th(x1,y1).nuTriangle;

fespace Vh(Th,P2BR);

Vh [a1,a2],[b1,b2],[c1,c2];

for (int i=0;i<Vh.ndofK;++i)
cout << i << " " << Vh(0,i) << endl;
for (int i=0;i<Vh.ndofK;++i)
{
  a1[]=0;
  int j=Vh(it1,i);
  a1[][j]=1; //      a bascis functions
  plot([a1,a2], wait=1);

  [b1,b2]=[a1,a2]; //      do the interpolation

  c1[] = a1[] - b1[];

  cout << " -----" << i << " " << c1[].max << " " << c1[].min << endl;
  cout << " a = " << a1[] << endl;
  cout << " b = " << b1[] << endl;
  assert(c1[].max < 1e-9 && c1[].min > -1e-9); //      check if the
  interpolation is correct

  //      check the derivative and numerical derivative

  cout << " dx(a1)(x1,y1) = " << dx(a1)(x1,y1) << " == " << DD(a1,h,0) << endl;
  assert( abs(dx(a1)(x1,y1)-DD(a1,h,0)) < 1e-5);
  assert( abs(dx(a2)(x1,y1)-DD(a2,h,0)) < 1e-5);
  assert( abs(dy(a1)(x1,y1)-DD(a1,0,h)) < 1e-5);
  assert( abs(dy(a2)(x1,y1)-DD(a2,0,h)) < 1e-5);
}

```

A real example using this finite element, just a small modification of the NSP2P1.edp examples, just the beginning is change to

```
load "BernadiRaugel"

real s0=clock();
mesh Th=square(10,10);
fespace Vh2(Th,P2BR);
fespace Vh(Th,P0);
Vh2 [u1,u2],[up1,up2];
Vh2 [v1,v2];
```

And the plot instruction is also changed because the pressure is constant, and we cannot plot isovalues of piecewise constant functions.

The Morley Element See the example bilapMorley.edp.

C.5 Add a new sparse solver

Warning the sparse solver interface as been completely rewritten in version 3.2 , so the section is obsolete, the example in are correct/

Only a fast sketch of the code is given here; for details see the .cpp code from SuperLU.cpp or NewSolve.cpp.

First the include files:

```
#include <iostream>
using namespace std;

#include "rgraph.hpp"
#include "error.hpp"
#include "AFunction.hpp"

// #include "lex.hpp"

#include "MatriceCreuse_tpl.hpp"
#include "slu_ddefs.h"
#include "slu_zdefs.h"
```

A small template driver to unified the double and Complex version.

```
template <class R> struct SuperLUDriver
{

};

template <> struct SuperLUDriver<double>
{
.... double version
};

template <> struct SuperLUDriver<Complex>
{
.... Complex version
```



```
};
```

To get Matrix value, we have just to remark that the Morse Matrice the storage, is the SLU_NR format is the compressed row storage, this is the transpose of the compressed column storage.

So if AA is a MatriceMorse you have with SuperLU notation.

```
n=AA.n;
m=AA.m;
nnz=AA.nbcoef;
a=AA.a;
asub=AA.cl;
xa=AA.lg;
options.Trans = TRANS;

Dtype_t R_SLU = SuperLUDriver<R>::R_SLU_T();
Create_CompCol_Matrix(&A, m, n, nnz, a, asub, xa, SLU_NC, R_SLU, SLU_GE);
```

To get vector infomation, to solver the linear solver $x = A^{-1}b$

```
void Solver(const MatriceMorse<R> &AA,KN<R> &x,const KN<R> &b) const
{
....
Create_Dense_Matrix(&B, m, 1, b, m, SLU_DN, R_SLU, SLU_GE);
Create_Dense_Matrix(&X, m, 1, x, m, SLU_DN, R_SLU, SLU_GE);
....
}
```

The two BuildSolverSuperLU function, to change the default sparse solver variable DefSparseSolver<double>::solver

```
MatriceMorse<double>::VirtualSolver *
BuildSolverSuperLU(DCL_ARG_SPARSE_SOLVER(double,A))
{
    if(verbosity>9)
        cout << " BuildSolverSuperLU<double>" << endl;
    return new SolveSuperLU<double>(*A,ds.strategy,ds.tgv,ds.epsilon,ds.tol_pivot,ds.tol_
}

MatriceMorse<Complex>::VirtualSolver *
BuildSolverSuperLU(DCL_ARG_SPARSE_SOLVER(Complex,A))
{
    if(verbosity>9)
        cout << " BuildSolverSuperLU<Complex>" << endl;
    return new SolveSuperLU<Complex>(*A,ds.strategy,ds.tgv,ds.epsilon,ds.tol_pivot,ds.tol_
}
```

The link to FreeFem++

```
class Init { public:
    Init();
};
```

To set the 2 default sparse solver double and complex:

```
DefSparseSolver<double>::SparseMatSolver SparseMatSolver_R ; ;
DefSparseSolver<Complex>::SparseMatSolver SparseMatSolver_C;
```

To save the default solver type

```
TypeSolveMat::TSolveMat TypeSolveMatdefaultvalue=TypeSolveMat::defaultvalue;
```

To reset to the default solver, call this function:

```
bool SetDefault()
{
    if(verbosity>1)
        cout << " SetDefault sparse to default" << endl;
    DefSparseSolver<double>::solver =SparseMatSolver_R;
    DefSparseSolver<Complex>::solver =SparseMatSolver_C;
    TypeSolveMat::defaultvalue =TypeSolveMat::SparseSolver;
}
```

To set the default solver to superLU, call this function:

```
bool SetSuperLU()
{
    if(verbosity>1)
        cout << " SetDefault sparse solver to SuperLU" << endl;
    DefSparseSolver<double>::solver =BuildSolverSuperLU;
    DefSparseSolver<Complex>::solver =BuildSolverSuperLU;
    TypeSolveMat::defaultvalue =TypeSolveMatdefaultvalue;
}
```

To add new function/name defaultsolver, defaultttoSuperLUin FreeFem++ , and set the default solver to the new solver., just do:

```
void init()
{
    SparseMatSolver_R= DefSparseSolver<double>::solver;
    SparseMatSolver_C= DefSparseSolver<Complex>::solver;

    if(verbosity>1)
        cout << "\n Add: SuperLU, defaultsolver defaultsolverSuperLU" << endl;
    TypeSolveMat::defaultvalue=TypeSolveMat::SparseSolver;
    DefSparseSolver<double>::solver =BuildSolverSuperLU;
    DefSparseSolver<Complex>::solver =BuildSolverSuperLU;
    // test if the name "defaultsolver" exist in freefem++
    if(! Global.Find("defaultsolver").NotNull() )
        Global.Add("defaultsolver", "(",new OneOperator0<bool>(SetDefault));
    Global.Add("defaultttoSuperLU", "(",new OneOperator0<bool>(SetSuperLU));
}

LOADFUNC(init);
```

To compile superlu.cpp, just do:

1. download the SuperLu 3.0 package and do

```
curl http://crd.lbl.gov/~xiaoye/SuperLU/superlu_3.0.tar.gz -o superlu_3.0.tar.gz
tar xvfz superlu_3.0.tar.gz
go SuperLU_3.0 directory
$EDITOR make.inc
make
```

2. In directoy include do to have a correct version of SuperLu header due to mistake in case of inclusion of double and Complex version in the same file.

```
tar xvfz ../SuperLU_3.0-include-ff.tar.gz
```

I will give a correct one to compile with freefm++.

To compile the FreeFem++ load file of SuperLu with freefem do: some find like :

```
ff-c++ SuperLU.cpp -L$HOME/work/LinearSolver/SuperLU_3.0/ -lsuperlu_3.0
```

And to test the simple example:

A example:

```
load "SuperLU"
verbosity=2;
for(int i=0;i<3;++i)
{
  // if i == 0 then SuperLu solver
  // i == 1 then GMRES solver
  // i == 2 then Default solver
  {
    matrix A =
      [[ 0, 1, 0, 10],
       [ 0, 0, 2, 0],
       [ 0, 0, 0, 3],
       [ 4,0 , 0, 0]];
    real[int] xx = [ 4,1,2,3], x(4), b(4);
    b = A*xx;
    cout << b << " " << xx << endl;
    set(A,solver=sparsesolver);
    x = A^-1*b;
    cout << x << endl;
  }

  {
    matrix<complex> A =
      [[ 0, 1i, 0, 10],
       [ 0 , 0, 2i, 0],
       [ 0, 0, 0, 3i],
       [ 4i,0 , 0, 0]];
    complex[int] xx = [ 4i,1i,2i,3i], x(4), b(4);
```

```
b = A*xx;
cout << b << " " << xx << endl;
set(A,solver=sparsesolver);
x = A^-1*b;
cout << x << endl;
}
if(i==0)defaulttoGMRES();
if(i==1)defaultsolver();
}
```

To Test do for exemple:

FreeFem++ SuperLu.edp

FreeFem++ LGPL License

This is The FreeFem++ software. Programs in it were maintained by

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- Jacques Morice <morice@ann.jussieu.fr>

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END OF TERMS AND CONDITIONS

Appendix D

Keywords

Main Keywords

adaptmesh
Cmatrix
R3
bool
border
break
buildmesh
catch
cin
complex
continue
cout
element
else
end
fespace
for
func
if
ifstream
include
int
intalledge
load
macro
matrix
mesh
movemesh
ofstream
plot
problem
real
return
savemesh
solve
string
try
throw
vertex
varf

while

Second category of Keywords

int1d
int2d
on
square

Third category of Keywords

dx
dy
convect
jump
mean

Fourth category of Keywords

wait
ps
solver
CG
LU
UMFPACK
factorize
init
endl

Other Reserved Words

x, y, z, pi, i,
sin, cos, tan, atan, asin, acos,
cotan, sinh, cosh, tanh, cotanh,
exp, log, log10, sqrt
abs, max, min,

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Book Description

Fruit of a long maturing process freefem, in its last avatar, `FreeFem++`, is a high level integrated development environment (IDE) for partial differential equations (PDE). It is the ideal tool for teaching the finite element method but it is also perfect for research to quickly test new ideas or multi-physics and complex applications.

`FreeFem++` has an advanced automatic mesh generator, capable of a posteriori mesh adaptation; it has a general purpose elliptic solver interfaced with fast algorithms such as the multi-frontal method UMFPACK. Hyperbolic and parabolic problems are solved by iterative algorithms prescribed by the user with the high level language of `FreeFem++`. It has several triangular finite elements, including discontinuous elements. Finally everything is there in `FreeFem++` to prepare research quality reports: color display online with zooming and other features and postscript printouts.

This book is ideal for students at Master level, for researchers at any level and for engineers also in financial mathematics.

Editorial Reviews

"... Impossible to put the book down, suspense right up to the last page..."

A. TANH, Siam Chronicle.

"... The chapter on discontinuous fems is so hilarious ..."

B. GALERKINE, Российской академии наук .